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Melnikov theory for finite-time vector fields

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Abstract. Melnikov theory provides a powerful tool for analysing time-dependent perturbations of autonomous vector fields that exhibit heteroclinic orbits. The standard theory requires that the perturbed vector field be defined, and bounded, for all times. In this paper, Melnikov theory is adapted so that it is applicable to vector fields that are defined over sufficiently large, but finite, time intervals. Such an extension is desirable when investigating Lagrangian trajectories in fluid flows under the effect of viscous perturbations; the resulting velocity field can only be guaranteed to be close to the unperturbed velocity field, corresponding to the inviscid limit, for finite times. Applications to transport in the viscous barotropic vorticity equation are given.

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1. Introduction

Melnikov theory has been developed to predict the splitting of homoclinic or heteroclinic orbits under non-autonomous perturbations. In particular, it can be used to establish the existence, or non-existence, of transverse homoclinic orbits in dynamical systems upon adding small non-autonomous terms to the governing vector field. Transverse homoclinic orbits, in turn, imply the existence of horseshoes, and therefore of chaotic dynamics. These results as well as more background and further references can be found, for instance, in the textbooks [3, 5]. Standard Melnikov theory is applicable to equations of the form

$$\dot{u} = f(u) + \varepsilon h(t, u; \varepsilon) \qquad u \in \mathbb{R}^n \tag{1.1}$$

where ε is small and $h(t, u; \varepsilon)$ is a nonlinearity that is defined for all times $t \in \mathbb{R}$. If $\bar{u}(t)$ denotes a homoclinic orbit of (1.1) to a certain hyperbolic equilibrium A_0, then the associated Melnikov integral that measures the splitting distance between stable and unstable manifolds of A_0 near the point $\bar{u}(0)$ upon varying ε is given by

$$d(\tau, \varepsilon) = \varepsilon \int_{-\infty}^{\infty} \langle \varphi(t), h(t + \tau, \bar{u}(t); 0)\rangle \, \mathrm{d}t + \mathrm{O}(\varepsilon^2). \tag{1.2}$$

Here, τ is the initial time for which we start solving (1.1), and $\varphi(t)$ is a certain non-zero bounded solution to

$$\dot{v} = -\mathrm{D}f(\bar{u}(t))^* v.$$

If the homoclinic orbit $\bar{u}(t)$, the bounded solution $\varphi(t)$ and the perturbation $h(t, u; \varepsilon)$ are known, we can compute the splitting distance up to terms of higher order and can then investigate the persistence of the homoclinic orbit $\bar{u}(t)$.

In order for Melnikov theory to work, it seems necessary that the perturbation be defined for all times t; otherwise, we cannot define the perturbed stable and unstable manifolds of the equilibrium A_0, and therefore cannot compute their distance upon varying ε. Indeed, stable and unstable manifolds are comprised, by their very definition, of solutions with a prescribed asymptotic behaviour as time tends to $\pm\infty$. The issue addressed in this paper is to give meaning to the concept of splitting distances for perturbations that are not defined for all times but are only given over sufficiently large, but finite, intervals.

Before we outline our approach, we comment on why such an extension might be desirable. Our motivation derived from an effort to understand the effect of viscous dissipation on two-dimensional vorticity-conserving flows in the oceanic context. The potential vorticity $q(x, y, t)$ of the fluid satisfies the partial differential equation (PDE)

$$\frac{\partial q}{\partial t} + \{\psi, q\} = \varepsilon\,[\Delta q + f(x, y, t)] \tag{1.3}$$

where

$$\{u, v\} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The stream function ψ of the fluid is related to the potential vorticity q by

$$q = \Delta\psi + \beta y.$$

The term involving β accounts for the Coriolis force. The dynamics of interest come from integrating these PDEs in a relevant domain (to avoid the complication of boundary effects, we usually think of the domain as being the plane with doubly periodic boundary conditions). The streamfunction gives the Lagrangian dynamics, i.e. particle trajectories, through the ordinary differential equation (ODE)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J\nabla\psi(x, y, t) \tag{1.4}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.5}$$

and ∇ is the gradient acting on the spatial variables (x, y). The position of a fluid particle is thus given as $(x(t), y(t))$ by further solving (1.4) with appropriate initial conditions. Due to the presence of the small positive parameter ε governing viscosity and forcing, the streamfunction, and hence the vector field, depends on that parameter.

In the limit $\varepsilon = 0$, vorticity is conserved in the PDE (1.3). This has significant implications for the ODE (1.4). In fact, it can then, in a sense, be viewed as integrable, see [2], and it follows that stable and unstable manifolds cannot intersect transversely—they either do not intersect or have branches coinciding. Consequently, the dynamical system (1.4) behaves as if it were autonomous. If it exhibits homoclinic loops or heteroclinic cycles, then these trajectories separate regions inside the fluid from the ambient fluid and prevent transport of fluid particles across these separatrices. If viscosity is taken into account, so that ε is positive, then these loops have the potential to break up and to form transverse intersections. This would imply that the formerly separated regions inside and outside of the loops can exchange fluid particles, and transport occurs (see [8]). Whether or not such (chaotic) transport occurs depends upon whether or not the loops intersect transversely upon adding the perturbation; this can be checked by calculating the associated Melnikov integral that appears in (1.2).

There are two issues that we need to address in order to compute the Melnikov integral. First, we need to know the perturbation $h(t, u; \varepsilon)$ to evaluate (1.2). This perturbation, however, is given by the velocity field that is only implicitly defined as a solution to the perturbed PDE (1.3). In [1], we showed that the Melnikov integral can nevertheless be calculated explicitly from information that pertains to the $\varepsilon = 0$ limit of (1.1). A particularly interesting consequence of the results in [1] is that, under a number of realistic circumstances, the manifolds split and do not intersect at all. This effect creates a channel through which fluid can flow from one region to another (but not vice versa) and precludes chaotic effects.

The Melnikov analysis in [1], however, requires that the perturbed velocity field be defined for all times and that it always remains close to the unperturbed velocity field. More realistically, however, we can only guarantee that the flows remain close over finite time periods. For instance, the PDE (1.3) is parabolic for positive ε so that we cannot expect that the solution exists for negative times. Thus, we are naturally led to investigate the breaking of separatrices under non-autonomous perturbations that are given only over a finite time interval.

Having commented on what motivated us to study Melnikov theory for finite-time perturbations, we shall outline our approach. Suppose then that the perturbation $h(t, u; \varepsilon)$ is given only over a certain finite but large time interval: what we will need is that the time interval contains at least the interval $(-C|\ln\varepsilon|, C|\ln\varepsilon|)$ as ε tends to zero for some positive constant C. In the first step, we investigate the persistence of stable and unstable manifolds under perturbation. Our strategy is to artificially extend the vector field outside of its time range of definition. There is no canonical way of carrying out this extension; we simply require that the extended vector field is $\mathrm{O}(\varepsilon^\nu)$-close to the unperturbed vector field for all times, where $\nu \in (\frac{1}{2}, 1)$ is fixed. Standard theory then provides us with invariant manifolds for the perturbed equation. These manifolds, however, depend upon the way in which we have extended the vector field outside its original domain of definition. We prove that any such extension leads to the same manifolds except for an error that is of the order $\mathrm{O}(\varepsilon^{2\nu}) = \mathrm{o}(\varepsilon)$. In fact, the larger the time range of definition of the original vector field, the closer are the invariant manifolds for different extensions: if, for instance, the length of the interval is polynomial in ε, then the manifolds are exponentially close to each other. We therefore refer to any such invariant manifolds as 'the' stable and unstable manifolds, keeping in mind that they are unique only up to terms of order $\mathrm{o}(\varepsilon)$. In the second step, we calculate the distance between the stable and unstable manifolds. We establish that the Melnikov integral, computed over the time range of definition, gives the $\mathrm{O}(\varepsilon)$ separation distance up to errors that are of order $\mathrm{o}(\varepsilon)$; note that this result is meaningful as the ambiguity in the construction of stable and unstable manifolds contributes only terms of order $\mathrm{o}(\varepsilon)$. In particular, we can discuss the nature of intersections of the invariant manifolds upon varying ε and investigate the existence of transverse intersections. It should be possible, for instance, to prove the existence of sets on which the flow behaves essentially as a shift on two symbols for a large but finite number of iterations, provided the intersections are transverse.

Related results have also recently been obtained, independently, by Haller and Poje [6] on invariant manifolds in finite-time vector fields. Their approach, in contrast to ours, is tailored to situations where the time dependence is relatively weak. In particular, their goal has been to compare the dynamics of the non-autonomous vector field with that of the frozen equations where the explicit time dependence of the vector field is neglected. In contrast, we allow a strong time dependence but assume that the finite-time vector field is close to one that is defined for all time.

The paper is organized as follows. We begin in section 2 by proving the existence of appropriate stable and unstable manifolds for finite-time perturbations, while section 3 is concerned with the calculation of their splitting distances using Melnikov integrals. In section 4 it is shown that the Melnikov integral can be calculated in the context of the viscous barotropic vorticity equation. Finally, in section 5, we apply the results to an explicit Rossby-wave solution of the viscous barotropic vorticity equation.

2. Persistence of invariant manifolds under finite-time perturbations

We formulate the problem by considering an unperturbed ODE for $u \in \mathbb{R}^n$ of the form

$$\dot{u} = f(u) \qquad u \in \mathbb{R}^n \tag{2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^k for some $k \geqslant 2$. The velocity field is denoted by f.

We assume that (2.1) has a solution $\bar{u}(t)$ that is bounded for $t \in \mathbb{R}$. The linearization

$$\dot{v} = \mathrm{D}f(\bar{u}(t))v \tag{2.2}$$

of (2.1) about the bounded trajectory $\bar{u}(t)$ then describes the behaviour of (2.1) near $\bar{u}(t)$. The evolution of (2.2) is denoted by $\Phi(t, s)$. Equation (2.2) is said to have an *exponential dichotomy* for $t \in \mathbb{R}^+$ if the following is true: there are constants $K \geqslant 1$ and $\theta > 0$, and a continuous family $P^{\mathrm{s}}(t)$ of projections, defined for $t \in \mathbb{R}^+$, such that

$$\begin{aligned} |\Phi(t,s)P^{\mathrm{s}}(s)| &\leqslant K\mathrm{e}^{-\theta|t-s|} \qquad &\text{for} \quad t \geqslant s \geqslant 0 \\ |\Phi(t,s)(\mathrm{id} - P^{\mathrm{s}}(s))| &\leqslant K\mathrm{e}^{-\theta|t-s|} \qquad &\text{for} \quad s \geqslant t \geqslant 0 \end{aligned} \tag{2.3}$$

and

$$\Phi(t,s)P^{\mathrm{s}}(s) = P^{\mathrm{s}}(t)\Phi(t,s) \qquad \text{for} \quad t, s \geqslant 0. \tag{2.4}$$

In other words, solutions associated with initial values in $\mathrm{R}(P^{\mathrm{s}}(s))$ decay exponentially in forward time, i.e. as t increases with $t > s$, while solutions belonging to initial values in $\mathrm{R}(\mathrm{id} - P^{\mathrm{s}}(s))$ decay exponentially in backward time, i.e. as t decreases with $s > t > 0$. We set $P^{\mathrm{u}}(t) = \mathrm{id} - P^{\mathrm{s}}(t)$.

Hypothesis 1. *The linearization (2.2) of (2.1) about $\bar{u}(t)$ has an exponential dichotomy on $\mathbb{R}^+$.*

It is then a well known consequence that there exists a unique local stable manifold $W_0^{\mathrm{s}}(\bar{u}) \in \mathbb{R}^n$ such that any solution $u(t)$ to (2.1) with $u(0)$ close to $\bar{u}(0)$ and $u(0) \in W_0^{\mathrm{s}}$ decays towards $\bar{u}(t)$ as $t \to \infty$ and satisfies $u(t) \in W_0^{\mathrm{s}}$ for all $t \geqslant 0$.

The issue addressed in this section is the behaviour of the stable manifold under perturbations. Of course, for small and smooth perturbations it is well known that the stable manifold persists. Here, the emphasis is on perturbations that are only given on large, but finite, time intervals. Stable manifolds, however, are defined by the behaviour of solutions as time tends to infinity.

The perturbations considered here are of the following form.

Hypothesis 2. *Let $h(t, u; \varepsilon)$ be a function defined for every $\varepsilon \in [0, \varepsilon_0)$ so that there are functions $\tau_\pm(\varepsilon)$ and constants $C > 0$ and $\nu \in (\frac{1}{2}, 1]$ such that*

- $h(\cdot, \cdot; \varepsilon) : [\tau_-(\varepsilon) - \frac{2\nu}{\theta}|\ln\varepsilon|, \tau_+(\varepsilon) + \frac{2\nu}{\theta}|\ln\varepsilon|] \times \mathbb{R}^n \to \mathbb{R}^n$ *is C^k in (t, u) for every fixed $\varepsilon \in [0, \varepsilon_0)$, and*
- $|h(t+\tau, u; \varepsilon)| + |\mathrm{D}_u h(t+\tau, u; \varepsilon)| \leqslant C\varepsilon^\nu$ *for $\tau \in [\tau_-(\varepsilon), \tau_+(\varepsilon)]$, $|t| \leqslant \frac{2\nu}{\theta}|\ln\varepsilon|$ and $u \in \mathbb{R}^n$.*

As mentioned above, stable manifolds are meaningful only for vector fields that are defined for all positive times. We then choose a function $h_1(t, u; \varepsilon)$ such that the following conditions are met:

$$\begin{aligned} &\text{(a)} \quad h_1 : \mathbb{R} \times \mathbb{R}^n \times [0, \varepsilon_0) \to \mathbb{R}^n, \ (t, u, \varepsilon) \mapsto h_1(t, u; \varepsilon) \text{ is } C^k \text{ in } (t, u) \text{ for every fixed } \varepsilon \\ &\text{(b)} \quad |h_1(t, u; \varepsilon)| + |\mathrm{D}_u h_1(t, u; \varepsilon)| \leqslant C\varepsilon^\nu \text{ for } (t, u) \in \mathbb{R} \times \mathbb{R}^n \\ &\text{(c)} \quad h_1(t + \tau, u; \varepsilon) = h(t + \tau, u; \varepsilon) \text{ for } \tau \in [\tau_-(\varepsilon), \tau_+(\varepsilon)], \ |t| \leqslant \frac{2\nu}{\theta}|\ln \varepsilon| \text{ and } u \in \mathbb{R}^n. \end{aligned} \tag{2.5}$$

Such a choice is clearly possible by using smooth cut-off functions. We consider the vector field

$$\dot{u} = f(u) + h_1(t + \tau, u; \varepsilon) \tag{2.6}$$

for $\tau \in [\tau_-(\varepsilon), \tau_+(\varepsilon)]$. We have introduced the parameter τ to account for the fact that the perturbation is non-autonomous. Note that the perturbation h_1 is defined for ε small and any positive t. Standard perturbation theory implies that the bounded solution $\bar{u}(t)$ persists as does its stable manifold $W_0^{\mathrm{s}}(\bar{u})$. Indeed, all solutions $u(t)$ that stay near $\bar{u}(t)$ for all positive times are captured by the following approach. Let

$$u(t) = \bar{u}(t) + v(t)$$

then $u(t)$ satisfies (2.6) and is close to $\bar{u}(t)$ for all positive times if and only if $v(t)$ satisfies the integral equation

$$v(t) = \Phi(t, 0)v_0 + \int_0^t \Phi(t, s)P^{\mathrm{s}}(s)G_1(s, v(s); \varepsilon)\,\mathrm{d}s + \int_\infty^t \Phi(t, s)P^{\mathrm{u}}(s)G_1(s, v(s); \varepsilon)\,\mathrm{d}s \tag{2.7}$$

for some $v_0 \in \mathrm{R}(P^{\mathrm{s}}(0))$, where the nonlinearity G_1 is given by

$$G_1(t, v; \varepsilon) = h_1(t + \tau, \bar{u}(t) + v; \varepsilon) + f(\bar{u}(t) + v) - f(\bar{u}(t)) - \mathrm{D}f(\bar{u}(t))v = \mathrm{O}(\varepsilon^\nu + |v|^2).$$

We suppress the dependence of G_1 on τ. Using the last estimate and an analogous estimate for the derivative $\mathrm{D}_v G_1$ of G_1, it is straightforward to solve the integral equation (2.7) for any given $v_0 \in \mathrm{R}(P^{\mathrm{s}}(0))$ with $|v_0|$ small by employing Banach's fixed-point theorem.

As a result, we obtain an invariant stable manifold $W_1^{\mathrm{s}}(\tau)$ near $\bar{u}(t)$ for (2.6). Originally, however, we were interested in the equation

$$\dot{u} = f(u) + h(t + \tau, u; \varepsilon). \tag{2.8}$$

Since $h(t + \tau, u; \varepsilon)$ and $h_1(t + \tau, u; \varepsilon)$ coincide for $0 \leqslant t \leqslant \frac{2\nu}{\theta}|\ln \varepsilon|$, we see that the manifold $W_1^{\mathrm{s}}(\tau)$ is invariant under the flow of (2.8) as long as $0 \leqslant t \leqslant \frac{2\nu}{\theta}|\ln \varepsilon|$. We may therefore regard $W_1^{\mathrm{s}}(\tau)$ as a 'stable' manifold for equation (2.8).

The interesting question, and indeed one of the main issues addressed in this paper, is the dependence of the stable manifold $W_1^{\mathrm{s}}(\tau)$ on the choice of the extension $h_1(t, u; \varepsilon)$. Thus, we shall derive estimates for the difference of two such manifolds $W_1^{\mathrm{s}}(\tau)$ and $W_2^{\mathrm{s}}(\tau)$ for two different extensions h_1 and h_2. We assume that both functions h_1 and h_2 satisfy the conditions (2.5) (a)–(c).

We decompose two solutions $u_j(t)$, $j = 1, 2$, to

$$\dot{u}_j = f(u_j) + h_j(t + \tau, u_j; \varepsilon)$$

as

$$u_j(t) = \bar{u}(t) + v_j(t).$$

If $u_j(t)$ stays close to $\bar{u}(t)$ for all positive times, then $v_j(t)$ satisfies the integral equation

$$v_j(t) = \Phi(t,0)v_0 + \int_0^t \Phi(t,s)P^{\mathrm{s}}(s)G_j(s, v_j(s); \varepsilon)\,\mathrm{d}s + \int_\infty^t \Phi(t,s)P^{\mathrm{u}}(s)G_j(s, v_j(s); \varepsilon)\,\mathrm{d}s \tag{2.9}$$

for some $v_0 \in \mathrm{R}(P^{\mathrm{s}}(0))$. The nonlinearities G_j are given by

$$G_j(t, v; \varepsilon) = h_j(t+\tau, \bar{u}(t) + v; \varepsilon) + f(\bar{u}(t) + v) - f(\bar{u}(t)) - \mathrm{D}f(\bar{u}(t))v = \mathrm{O}(\varepsilon^\nu + |v|^2).$$

As before, we obtain two invariant manifolds $W_1^{\mathrm{s}}(\tau)$ and $W_2^{\mathrm{s}}(\tau)$ formed by solutions to (2.9) for varying $v_0 \in \mathrm{R}(P^{\mathrm{s}}(0))$.

We remark that the solutions $v_j(t)$ satisfy the estimate

$$|v_j(t)| \leqslant C(|v_0| + \varepsilon^\nu) \tag{2.10}$$

uniformly in $t \geqslant 0$. Here, and in the following, various different constants that are independent of ε, t and τ are denoted by C.

The distance between the manifolds $W_1^{\mathrm{s}}(\tau)$ and $W_2^{\mathrm{s}}(\tau)$ is measured at $t = 0$ in the direction of the complement $\mathrm{R}(P^{\mathrm{u}}(0))$. Assuming that $v_1(t)$ and $v_2(t)$ satisfy (2.9) for $j = 1, 2$ for the same value of v_0, we define

$$w(t) = v_1(t) - v_2(t).$$

The difference $w(t)$ then satisfies the equation

$$\begin{aligned} w(t) = &\int_0^t \Phi(t,s)P^{\mathrm{s}}(s)(G_1(s, v_1(s); \varepsilon) - G_2(s, v_2(s); \varepsilon))\,\mathrm{d}s \\ &+ \int_\infty^t \Phi(t,s)P^{\mathrm{u}}(s)(G_1(s, v_1(s); \varepsilon) - G_2(s, v_2(s); \varepsilon))\,\mathrm{d}s. \end{aligned} \tag{2.11}$$

We next estimate the difference $G_1(s, v_1; \varepsilon) - G_2(s, v_2; \varepsilon)$. We obtain

$$\begin{aligned} &|f(\bar{u}(t) + v_1) - f(\bar{u}(t) + v_2) - \mathrm{D}f(\bar{u}(t))(v_1 - v_2)| \\ &\qquad = \left| \int_0^1 (\mathrm{D}f(\bar{u}(t) + v_2 + s(v_1 - v_2)) - \mathrm{D}f(\bar{u}(t)))\,\mathrm{d}s\,(v_1 - v_2) \right| \\ &\qquad \leqslant C(|v_1| + |v_2|)|v_1 - v_2| \\ &\qquad \leqslant C(|v_0| + \varepsilon^\nu)|w| \end{aligned}$$

since $|v_1| + |v_2| \leqslant C(|v_0| + \varepsilon^\nu)$. Furthermore, if we restrict to $0 \leqslant t \leqslant \frac{2\nu}{\theta}|\ln\varepsilon|$ and $\tau \in [\tau_-(\varepsilon), \tau_+(\varepsilon)]$, then the functions h_1 and h_2 coincide, and we obtain

$$\begin{aligned} &|h_1(t+\tau, \bar{u}(t) + v_1; \varepsilon) - h_2(t+\tau, \bar{u}(t) + v_2; \varepsilon)| \\ &\qquad = |h(t+\tau, \bar{u}(t) + v_1; \varepsilon) - h(t+\tau, \bar{u}(t) + v_2; \varepsilon)| \leqslant C\varepsilon^\nu|w| \end{aligned}$$

for $0 \leqslant t \leqslant \frac{2\nu}{\theta}|\ln\varepsilon|$ and $\tau \in [\tau_-(\varepsilon), \tau_+(\varepsilon)]$. In summary, we have

$$|G_1(s, v_1; \varepsilon) - G_2(s, v_2; \varepsilon)| \leqslant \begin{cases} C(\varepsilon^\nu + |v_0|)|w| & \text{for} \quad 0 \leqslant t \leqslant \frac{2\nu}{\theta}|\ln\varepsilon| \\ C(\varepsilon^\nu + |v_0|\,|w|) & \text{for} \quad t \geqslant \frac{2\nu}{\theta}|\ln\varepsilon| \end{cases} \tag{2.12}$$

uniformly in $\tau \in [\tau_-(\varepsilon), \tau_+(\varepsilon)]$. After these preliminary calculations, we return to (2.11) and estimate this equation as follows:

$$\begin{aligned}|w(t)| &\leqslant \left|\int_0^t \Phi(t,s)P^{\mathrm{s}}(s)(G_1(s,v_1(s);\varepsilon) - G_2(s,v_2(s);\varepsilon))\,\mathrm{d}s\right| \\ &\quad + \left|\int_\infty^t \Phi(t,s)P^{\mathrm{u}}(s)(G_1(s,v_1(s);\varepsilon) - G_2(s,v_2(s);\varepsilon))\,\mathrm{d}s\right| \\ &\leqslant K\int_0^t \mathrm{e}^{-\theta(t-s)}|G_1(s,v_1(s);\varepsilon) - G_2(s,v_2(s);\varepsilon)|\,\mathrm{d}s \\ &\quad + K\int_\infty^t \mathrm{e}^{-\theta(s-t)}|G_1(s,v_1(s);\varepsilon) - G_2(s,v_2(s);\varepsilon)|\,\mathrm{d}s.\end{aligned}$$

We now restrict to $0 \leqslant t \leqslant \frac{2\nu}{\theta}|\ln\varepsilon|$. Using (2.12), we obtain

$$\begin{aligned}|w(t)| &\leqslant C(|v_0| + \varepsilon^\nu)\left(\int_0^t \mathrm{e}^{-\theta(t-s)}|w(s)|\,\mathrm{d}s + \int_{\frac{2\nu}{\theta}|\ln\varepsilon|}^t \mathrm{e}^{-\theta(s-t)}|w(s)|\,\mathrm{d}s\right) \\ &\quad + C\int_\infty^{\frac{2\nu}{\theta}|\ln\varepsilon|} \mathrm{e}^{-\theta(s-t)}(\varepsilon^\nu + |v_0|\,|w(s)|)\,\mathrm{d}s \\ &\leqslant C(|v_0| + \varepsilon^\nu)\left(\int_0^t \mathrm{e}^{-\theta(t-s)}|w(s)|\,\mathrm{d}s + \int_{\frac{2\nu}{\theta}|\ln\varepsilon|}^t \mathrm{e}^{-\theta(s-t)}|w(s)|\,\mathrm{d}s\right) \\ &\quad + C(\varepsilon^\nu + |v_0|(|v_0| + \varepsilon^\nu))\varepsilon^{2\nu}\mathrm{e}^{\theta t}.\end{aligned} \tag{2.13}$$

To obtain an optimal estimate, we introduce a weighted norm. Choose some number $\tilde\theta$ with $0 < \tilde\theta < \theta$. Set

$$\delta(\varepsilon) = \frac{2\nu}{\theta}|\ln\varepsilon|.$$

We then define

$$\|w(t)\| := \mathrm{e}^{\tilde\theta(\delta(\varepsilon)-t)}|w(t)|.$$

Multiplying (2.13) by $\exp[\tilde\theta(\delta(\varepsilon) - t)]$ and taking the norm on both sides, we obtain for $0 \leqslant t \leqslant \delta(\varepsilon)$:

$$\begin{aligned}\mathrm{e}^{\tilde\theta(\delta(\varepsilon)-t)}|w(t)| &\leqslant C\mathrm{e}^{\tilde\theta(\delta(\varepsilon)-t)}\left[(|v_0| + \varepsilon^\nu)\left(\int_0^t \mathrm{e}^{-\theta(t-s)}|w(s)|\,\mathrm{d}s + \int_{\delta(\varepsilon)}^t \mathrm{e}^{-\theta(s-t)}|w(s)|\,\mathrm{d}s\right)\right] \\ &\quad + C(\varepsilon^\nu + |v_0|(|v_0| + \varepsilon^\nu))\,\mathrm{e}^{(\theta-\tilde\theta)(t-\delta(\varepsilon))} \\ &\leqslant C\mathrm{e}^{\tilde\theta(\delta(\varepsilon)-t)}(|v_0| + \varepsilon^\nu)\sup_{0\leqslant s\leqslant\delta(\varepsilon)}\|w(s)\| \\ &\quad \times\left(\int_0^t \mathrm{e}^{-\theta(t-s)}\mathrm{e}^{-\tilde\theta(\delta(\varepsilon)-s)}\,\mathrm{d}s + \int_{\delta(\varepsilon)}^t \mathrm{e}^{-\theta(s-t)}\mathrm{e}^{-\tilde\theta(\delta(\varepsilon)-s)}\,\mathrm{d}s\right) + C(\varepsilon^\nu + |v_0|^2) \\ &\leqslant C(|v_0| + \varepsilon^\nu)\sup_{0\leqslant s\leqslant\delta(\varepsilon)}\|w(s)\|\,\mathrm{e}^{\tilde\theta(\delta(\varepsilon)-t)} \\ &\quad \times\left(\mathrm{e}^{-\tilde\theta(\delta(\varepsilon)-t)} + \mathrm{e}^{-\theta t}\mathrm{e}^{-\tilde\theta\delta(\varepsilon)} + \mathrm{e}^{-\tilde\theta(\delta(\varepsilon)-t)} + \mathrm{e}^{-\theta(\delta(\varepsilon)-t)}\right) + C(\varepsilon^\nu + |v_0|^2) \\ &\leqslant C(\varepsilon^\nu + |v_0|^2) + C(|v_0| + \varepsilon^\nu)\sup_{0\leqslant s\leqslant\delta(\varepsilon)}\|w(s)\|.\end{aligned}$$

Hence, we can conclude that

$$\sup_{0\leqslant t\leqslant \frac{2\nu}{\theta}|\ln\varepsilon|} \|w(t)\| \leqslant C(\varepsilon^{\nu} + |v_0|^2)$$

and therefore

$$|w(t)| \leqslant C(\varepsilon^{\nu} + |v_0|^2)\,\mathrm{e}^{-\tilde{\theta}(\frac{2\nu}{\theta}|\ln\varepsilon|-t)} = C(\varepsilon^{\nu} + |v_0|^2)\varepsilon^{\frac{2\nu\tilde{\theta}}{\theta}}\mathrm{e}^{\tilde{\theta}t} \tag{2.14}$$

for $0 \leqslant t \leqslant \frac{2\nu}{\theta}|\ln\varepsilon|$. Note that the solution $\bar{u}(t)$ was quite arbitrary: parametrizing along a different solution in $W_0^{\mathrm{s}}(\bar{u})$, if necessary, we may assume that $v_0 = 0$. Thus, the distance between solutions in $W_1^{\mathrm{s}}(\tau)$ and $W_2^{\mathrm{s}}(\tau)$ measured orthogonally to the tangent space of W_0^{s} can be estimated by

$$|w(t)| \leqslant C\varepsilon^{\nu}\varepsilon^{\frac{2\nu\tilde{\theta}}{\theta}}\mathrm{e}^{\tilde{\theta}t} \tag{2.15}$$

for $0 \leqslant t \leqslant \frac{2\nu}{\theta}|\ln\varepsilon|$. In summary, we have proved the following theorem.

Theorem 1. *Assume that hypotheses 1 and 2 are met. For any two functions h_1 and h_2 that satisfy (2.5), we have the following results: take any two solutions $u_1(t)$ in $W_1^{\mathrm{s}}(\tau)$ and $u_2(t)$ in $W_2^{\mathrm{s}}(\tau)$ close to $\bar{u}(t)$ such that $u_1(0) - u_2(0)$ is contained in* $\mathrm{R}(P^{\mathrm{u}}(0))$. *Their difference can then be estimated by*

$$|u_1(0) - u_2(0)| \leqslant C\varepsilon^{2\nu}$$

uniformly for $\tau \in [\tau_-(\varepsilon), \tau_+(\varepsilon)]$.

Proof. If $u_1(0) - u_2(0)$ is contained in $\mathrm{R}(P^{\mathrm{u}}(0))$, there exists a unique element $\tilde{u}_0$ in the stable manifold W_0^{s} such that $u_j(0) = \tilde{u}_0 + v_j^{\mathrm{u}}$ for certain elements $v_j^{\mathrm{u}} \in \mathrm{R}(P^{\mathrm{u}}(0))$ with $j = 1, 2$. By the roughness theorem for exponential dichotomies [4], the linearization of (2.2) about $\tilde{u}(t)$ also has an exponential dichotomy with the same null space $\mathrm{R}(P^{\mathrm{u}}(0))$ at $t = 0$ (see [4]). Therefore, the arguments given above apply with $v_0 = 0$, and the theorem follows from (2.15) upon choosing $\tilde{\theta}$ so that $\frac{1}{2}\theta \leqslant \tilde{\theta} \leqslant \theta$. □

In other words, any two choices of h_1 and h_2 lead to invariant stable manifolds whose distance, at $t = 0$, is less than $C\varepsilon^{2\nu}$ uniformly in $\tau \in [\tau_-(\varepsilon), \tau_+(\varepsilon)]$.

3. Splitting of separatrices

We apply the results obtained in the last section to the splitting of separatrices in autonomous vector fields under finite-time perturbations. Consider the equation

$$\dot{u} = f(u) \qquad u \in \mathbb{R}^n \tag{3.1}$$

where $f(u)$ is C^k for some $k \geqslant 2$.

Hypothesis 3. *Equation (3.1) has hyperbolic equilibria A_0 and B_0, and there is a heteroclinic trajectory $\bar{u}(t)$ which connects A_0 to B_0. Furthermore, we assume that*

$$T_{\bar{u}(0)}W^{\mathrm{u}}(A_0) \cap T_{\bar{u}(0)}W^{\mathrm{s}}(B_0) = \mathrm{span}\{\dot{\bar{u}}(0)\} \qquad \dim W^{\mathrm{u}}(A_0) + \dim W^{\mathrm{s}}(B_0) = n.$$

Since the equilibria are hyperbolic, the linearization of (3.1) about $\bar{u}(t)$ satisfies hypothesis 1 for positive and negative times. We are then interested in the fate of the heteroclinic orbit $\bar{u}(t)$ upon perturbing the equation to

$$\dot{u} = f(u) + h_1(t+\tau, u; \varepsilon) \qquad u \in \mathbb{R}^n \tag{3.2}$$

for some function h_1 that satisfies (2.5).

We can now employ theorem 1 and conclude that there are stable and unstable manifolds $W^{\mathrm{s}}(\tau; B_0)$ and $W^{\mathrm{u}}(\tau; A_0)$ of (3.2) for any small ε and $\tau \in [\tau_-(\varepsilon), \tau_+(\varepsilon)]$. The distance between any two stable (or unstable) manifolds obtained for different functions h_1 is less than $C\varepsilon^{2\nu}$ outside a small neighbourhood of the equilibria. We are interested in the distance between the stable and unstable manifolds, which we would like to measure by a Melnikov integral.

Before stating the result, we need one more piece of information. As a consequence of hypothesis 3, the adjoint variational equation

$$\dot{v} = -\mathrm{D}f(\bar{u}(t))^* v \tag{3.3}$$

along the heteroclinic trajectory $\bar{u}(t)$ has a unique, up to constant multiples, non-zero bounded solution $\varphi(t)$, and this solution satisfies

$$|\varphi(t)| \leqslant C|\varphi(0)|\,\mathrm{e}^{-\theta|t|}$$

for $t \in \mathbb{R}$ (see, for instance, [9]). We then have the following result.

Lemma 1. *Suppose that hypotheses 2 and 3 are true. The stable and unstable manifolds $W^{\mathrm{s}}(\tau; B_0)$ and $W^{\mathrm{u}}(\tau; A_0)$ of (3.2) with $\tau \in [\tau_-(\varepsilon), \tau_+(\varepsilon)]$ have an intersection near $\bar{u}(0)$ if and only if*

$$d(\tau, \varepsilon) = \int_{-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\frac{2\nu}{\theta}|\ln\varepsilon|} \langle \varphi(t), h(t+\tau, \bar{u}(t); \varepsilon)\rangle\,\mathrm{d}t + \mathrm{O}(\varepsilon^{2\nu}) = 0. \tag{3.4}$$

Moreover, the intersection is transverse if, and only if, $\frac{\partial}{\partial\tau}d(\tau, \varepsilon) \neq 0$.

The distance function $d(\tau, \epsilon)$ in the above lemma is not normalized: a different choice of the function $\varphi(t)$ results in a different distance function. To make the distance function $d(\tau, \epsilon)$ unique, we could either normalize $\varphi(t)$ so that $|\varphi(0)| = 1$ or else divide the expression on the right-hand side of (3.4) by $|\varphi(0)|$. For simplicity, we refrain from normalizing the distance function in this fashion as we would only introduce a constant factor in front of the right-hand side of (3.4).

Proof. If $n = 2$ (i.e. for a two-dimensional phase space $\mathbb{R}^2$), it then follows from [1, theorem 1] and its proof that the distance function $d(\tau, \varepsilon)$ that describes the intersections of the stable and unstable manifolds is of the form

$$d(\tau, \varepsilon) = \int_{-\infty}^{\infty} \langle \varphi(t), G_1(t+\tau, v_1(t); \varepsilon)\rangle\,\mathrm{d}t \tag{3.5}$$

where $v_1 = v_1(\tau, \varepsilon)$ satisfies

$$\sup_{t\in\mathbb{R}} |v_1(\tau, \varepsilon)(t)| \leqslant C\varepsilon^{\nu}. \tag{3.6}$$

Recall that G_1 is given by

$$G_1(t, v; \varepsilon) = h_1(t, \bar{u}(t) + v; \varepsilon) + f(\bar{u}(t) + v) - f(\bar{u}(t)) - \mathrm{D}f(\bar{u}(t))v.$$

If $n > 2$, expression (3.5) for the distance function and the estimate (3.6) for v_1 are a consequence of [11, theorem 4] and its proof: it is straightforward to verify that the proof of [11, theorem 4] also works in the case where the perturbation h_1 satisfies (2.5). In particular, in the notation of [11], the $\mathrm{O}(\mu)$-estimates for the perturbation μH and the solution $w(t, \mu)$ can be replaced by $\mathrm{O}(\mu^\nu)$ estimates.

It remains to simplify (3.5). Exploiting (2.5) (b) and the estimate (3.6) for v_1, we obtain

$$d(\tau, \varepsilon) = \int_{-\infty}^{\infty} \langle \varphi(t), h_1(t + \tau, \bar{u}(t); \varepsilon)\rangle \,\mathrm{d}t + \mathrm{O}(\varepsilon^{2\nu}).$$

Finally, we truncate the interval of integration to $(-\frac{2\nu}{\theta}|\ln\varepsilon|, \frac{2\nu}{\theta}|\ln\varepsilon|)$; the resulting error is of order $\mathrm{O}(\varepsilon^{2\nu})$ since $\varphi(t)$ decays exponentially. Note that the functions h and h_1 coincide on the interval $(-\frac{2\nu}{\theta}|\ln\varepsilon|, \frac{2\nu}{\theta}|\ln\varepsilon|)$; see (2.5) (c). □

Theorem 1 and lemma 1 show that the splitting of separatrices is well defined even for finite-time perturbations. Indeed, any two stable and unstable manifolds differ by at most $\mathrm{O}(\varepsilon^{2\nu})$. On the other hand, the splitting distance is given by

$$d(\tau, \varepsilon) = \int_{-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\frac{2\nu}{\theta}|\ln\varepsilon|} \langle \varphi(t), h(t + \tau, \bar{u}(t); \varepsilon)\rangle \,\mathrm{d}t + \mathrm{O}(\varepsilon^{2\nu}).$$

Since $\nu > \frac{1}{2}$, the error term is of the form $\mathrm{O}(\varepsilon^{1+\gamma})$ for some $\gamma > 0$. Even if the integral itself is of the order ε, the error term would be of higher order. Hence, if

$$\left| \int_{-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\frac{2\nu}{\theta}|\ln\varepsilon|} \langle \varphi(t), h(t + \tau, \bar{u}(t); \varepsilon)\rangle \,\mathrm{d}t \right| \geqslant a\varepsilon$$

for some constant $a > 0$ that is independent of ε and τ, then the stable and unstable manifolds cannot intersect however we choose the extension h_1.

4. An application to the breaking of vorticity conservation by viscous dissipation

We return to the issue of transport in the viscous barotropic vorticity equation in two spatial dimensions (see section 1). Thus, assume that $\psi(x, y, t; \varepsilon)$ is a solution to

$$\frac{\partial q}{\partial t} + \{\psi, q\} = \varepsilon\,[\Delta q + f(x, y, t)] \tag{4.1}$$

where

$$q = \Delta\psi + \beta y \qquad \{\psi, q\} = \frac{\partial\psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial q}{\partial x}.$$

We restrict our attention to travelling-wave solutions to this equation. It has been shown in [1, lemma 6] that such waves always travel in the x-direction. Therefore, we assume that, for $\varepsilon = 0$, the solution $\psi(x, y, t; 0)$ is given by

$$\psi(x, y, t; 0) = \Psi^0(x - ct, y)$$

for an appropriate function $\Psi^0(\xi, \eta)$ and a certain wave speed c. For $\varepsilon = 0$, the Lagrangian dynamics in the moving frame $(\xi, \eta) = (x - ct, y)$ is then governed by the equation

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = J\nabla(\Psi^0(\xi, \eta) + c\eta) \tag{4.2}$$

which is a Hamiltonian with energy $\Psi^0(\xi, \eta) + c\eta$. The skew-symmetric matrix J has been defined in (1.5). We assume that (4.2) satisfies the following assumption.

Hypothesis 4. *Equation (4.2) has a homoclinic trajectory $\bar{u}(t) = (\bar{\xi}, \bar{\eta})(t)$ that connects the hyperbolic equilibrium (ξ_A, η_A) to itself.*

For $\varepsilon > 0$, we write

$$\Psi(\xi, \eta, t; \varepsilon) := \psi(\xi + ct, \eta, t; \varepsilon).$$

We shall then investigate the Lagrangian dynamics

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = J\nabla(\Psi(\xi, \eta, t; \varepsilon) + c\eta). \tag{4.3}$$

Note that, for $\varepsilon > 0$, the PDE (4.1) is parabolic so that, in general, the streamfunction $\Psi(\xi, \eta, t; \varepsilon)$ exists only for $t \geqslant 0$. We are therefore forced to consider finite-time perturbations of (4.2). We assume that the perturbation satisfies hypothesis 2.

Hypothesis 5. *The perturbation*

$$h(t, \xi, \eta; \varepsilon) := J\nabla(\Psi(\xi, \eta, t; \varepsilon) - \Psi^0(\xi, \eta))$$

is defined for every $\varepsilon \in [0, \varepsilon_0)$ so that, for some $\kappa > 0$,

- *$h(\cdot, \cdot; \varepsilon) : [0, 2\varepsilon^{-\kappa}] \times \mathbb{R}^2 \to \mathbb{R}^2$ is C^k in (t, ξ, η) for every fixed $\varepsilon \in [0, \varepsilon_0)$, and*
- *there exist constants $C > 0$ and $\nu \in (\frac{1}{2}, 1]$ so that $|h(t, \xi, \eta; \varepsilon)| + |\mathrm{D}_{(\xi,\eta)}h(t, \xi, \eta; \varepsilon)| \leqslant C\varepsilon^{\nu}$ for $0 \leqslant t \leqslant 2\varepsilon^{-\kappa}$ and $(\xi, \eta) \in \mathbb{R}^2$.*

Before we state the theorem, we define

$$Q^0(\xi, \eta) := \Delta(\Psi^0(\xi, \eta) + c\eta) + \beta\eta \qquad F(\xi, \eta, t) := f(\xi + ct, \eta, t).$$

The following modification of [1, theorem 1] holds.

Theorem 2. *Under the assumptions given above, the separation function for equation (4.3) has the form*

$$d(\tau, \varepsilon) = \varepsilon \int_{-\infty}^{\infty} [\Delta Q^0(\bar{\xi}(t), \bar{\eta}(t)) - \Delta Q^0(\xi_A, \eta_A)]\,\mathrm{d}t$$
$$+\varepsilon \int_{-\infty}^{\infty} [F(\bar{\xi}(t), \bar{\eta}(t), t + \tau) - F(\xi_A, \eta_A, t + \tau)]\,\mathrm{d}t + \mathrm{O}(\varepsilon^{2\nu}) \tag{4.4}$$

uniformly in $\tau \in \left[\frac{2\nu}{\theta}|\ln\varepsilon|, 2\varepsilon^{-\kappa} - \frac{2\nu}{\theta}|\ln\varepsilon|\right]$.

Note that, since $\nu > \frac{1}{2}$ by hypothesis 5, the error term in the above formula is of higher order than ε.

Proof. It is convenient to shift time. Thus, instead of considering the interval $(0, 2\varepsilon^{-\kappa})$, we transform the interval to $(-\varepsilon^{-\kappa}, \varepsilon^{-\kappa})$. Then, τ varies in the interval $\tau \in [-\varepsilon^{-\kappa} + \frac{2\nu}{\theta}|\ln\varepsilon|, \varepsilon^{-\kappa} - \frac{2\nu}{\theta}|\ln\varepsilon|]$, and we assume that hypothesis 5 is met for $t \in (-\varepsilon^{-\kappa}, \varepsilon^{-\kappa})$ rather than for $t \in (0, 2\varepsilon^{-\kappa})$.

We now apply lemma 1 to (4.3); this is possible due to our assumptions. The distance function is then given by

$$d(\tau,\varepsilon)=\int_{-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\frac{2\nu}{\theta}|\ln\varepsilon|}\Big\langle\varphi(t),\,J\nabla(\Psi(\bar u(t),t+\tau;\varepsilon)-\Psi^0(\bar u(t)))\Big\rangle\,\mathrm{d}t+\mathrm{O}(\varepsilon^{2\nu})$$

where $\varphi(t)$ is a non-zero bounded solution to the adjoint variational equation associated with (4.2) about $\bar u(t)$. It follows from [1, lemma 1 and theorem 1] that we can take $\varphi(t)=\nabla Q^0(\bar u(t))$. Thus,

$$d(\tau,\varepsilon)=\int_{-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\frac{2\nu}{\theta}|\ln\varepsilon|}\Big\langle\nabla Q^0(\bar u(t)),\,J\nabla(\Psi(\bar u(t),t+\tau;\varepsilon)-\Psi^0(\bar u(t)))\Big\rangle\,\mathrm{d}t+\mathrm{O}(\varepsilon^{2\nu}).$$

First, we transform into the original non-moving coordinates $(x,y)=(\xi+ct,\eta)$. In the original coordinates, the homoclinic solution $\bar u(t)$ and the equilibrium (ξ_A,η_A) are given by

$$\bar z(t;\tau)=(\bar\xi(t-\tau)+ct,\bar\eta(t-\tau))\qquad A_0(t)=(\xi_A+ct,\eta_A).$$

Define the functions ψ^0 and ψ^1 by

$$\psi^0(x,y,t):=\psi(x,y,t;\varepsilon)\qquad \varepsilon\psi^1(x,y,t;\varepsilon):=\psi(x,y,t;\varepsilon)-\psi^0(x,y,t;0)$$

and let $q^0=\Delta\psi^0+\beta y$ and $q^1=\Delta\psi^1$ so that $q=q^0+\varepsilon q^1$. On account of hypothesis 5, we then have

$$|\psi^1(x,y,t;\varepsilon)|+|\nabla\psi^1(x,y,t;\varepsilon)|+|q^1(x,y,t;\varepsilon)|\leqslant C\varepsilon^{-1+\nu}\tag{4.5}$$

uniformly in (x,y,t) and ε. Writing the distance function in the original coordinates and shifting time $t\mapsto t-\tau$, we obtain

$$\begin{aligned}d(\tau,\varepsilon)&=\varepsilon\int_{\tau-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\tau+\frac{2\nu}{\theta}|\ln\varepsilon|}\langle\nabla q^0(\bar z(t;\tau),t),\,J\nabla\psi^1(\bar z(t;\tau),t;\varepsilon)\rangle\,\mathrm{d}t+\mathrm{O}(\varepsilon^{2\nu})\\&=\varepsilon\int_{\tau-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\tau+\frac{2\nu}{\theta}|\ln\varepsilon|}\{\psi^1,q^0\}(\bar z(t;\tau),t;\varepsilon)\,\mathrm{d}t+\mathrm{O}(\varepsilon^{2\nu})\end{aligned}$$

see [1, equation (6.3)]. We write

$$d(\tau,\varepsilon)=\varepsilon M(\tau)+\mathrm{O}(\varepsilon^{2\nu})\tag{4.6}$$

where

$$M(\tau)=\int_{\tau-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\tau+\frac{2\nu}{\theta}|\ln\varepsilon|}\{\psi^1,q^0\}(\bar z(t;\tau),t;\varepsilon)\,\mathrm{d}t.\tag{4.7}$$

Throughout the remaining part of the proof, we closely follow the arguments in [1, pp 64–5] and refer to that paper for more details. The Melnikov integral in the above formula (4.7) can be calculated by applying the operator $\int_{\tau-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\tau+\frac{2\nu}{\theta}|\ln\varepsilon|}\mathrm{d}t$ to the equation [1, (6.7)],

$$\begin{aligned}\{\psi^1,q^0\}(\bar z(t;\tau),t)&=[\Delta q^0(\bar z(t;\tau),t)-\Delta q^0(A_0(t),t)]+[f(\bar z(t;\tau),t)-f(A_0(t),t)]\\&\quad+\left[\left\{\Delta q^0+f+\varepsilon\left[\Delta q^1-\{\psi^1,q^1\}\right]\right\}(A_0(t),t)-\frac{\mathrm{d}q^1}{\mathrm{d}t}(\bar z(t;\tau),t)\right]\\&\quad+\varepsilon\left[\left(\Delta q^1-\{\psi^1,q^1\}\right)(\bar z(t;\tau),t)-\left(\Delta q^1-\{\tilde\psi^1,q^1\}\right)(A_0(t),t)\right]\end{aligned}$$

that is satisfied on the finite interval $(\tau - \frac{2\nu}{\theta}|\ln\varepsilon|, \tau + \frac{2\nu}{\theta}|\ln\varepsilon|)$. The computations on [1, p 64] remain valid: the only difference is that the error term is of order $\mathrm{O}(\varepsilon^{-1+2\nu})$ instead of $\mathrm{O}(\varepsilon)$: this estimate follows as on [1, p 64] upon using the estimate (4.5). Furthermore, additional terms of order $\mathrm{O}(\varepsilon^{-1+2\nu})$ arise when the integral is evaluated at the finite limits $\tau \pm \frac{2\nu}{\theta}|\ln\varepsilon|$. In summary, we obtain the formula

$$M(\tau) = \int_{\tau-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\tau+\frac{2\nu}{\theta}|\ln\varepsilon|} [\Delta q^0(\bar{z}(t;\tau)) - \Delta q^0(A_0(t))]\,\mathrm{d}t + \int_{\tau-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\tau+\frac{2\nu}{\theta}|\ln\varepsilon|} [f(\bar{z}(t;\tau),t) - f(A_0(t),t)]\,\mathrm{d}t + \mathrm{O}(\varepsilon^{-1+2\nu}).$$

Switching back to the moving coordinates and shifting the time variable, $t \mapsto t + \tau$, we finally arrive at the expression

$$\begin{aligned} M(\tau) &= \int_{-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\frac{2\nu}{\theta}|\ln\varepsilon|} [\Delta Q^0(\bar{\xi}(t),\bar{\eta}(t)) - \Delta Q^0(\xi_A,\eta_A)]\,\mathrm{d}t \\ &\quad + \int_{-\frac{2\nu}{\theta}|\ln\varepsilon|}^{\frac{2\nu}{\theta}|\ln\varepsilon|} [F(\bar{\xi}(t),\bar{\eta}(t),t+\tau) - F(\xi_A,\eta_A,t+\tau)]\,\mathrm{d}t + \mathrm{O}(\varepsilon^{-1+2\nu}) \\ &= \int_{-\infty}^{\infty} [\Delta Q^0(\bar{\xi}(t),\bar{\eta}(t)) - \Delta Q^0(\xi_A,\eta_A)]\,\mathrm{d}t \\ &\quad + \int_{-\infty}^{\infty} [F(\bar{\xi}(t),\bar{\eta}(t),t+\tau) - F(\xi_A,\eta_A,t+\tau)]\,\mathrm{d}t + \mathrm{O}(\varepsilon^{-1+2\nu}). \end{aligned}$$

Upon substituting this term into (4.6), we see that the assertions of the theorem are true. □

We have observed that the unperturbed and perturbed streamfunction will certainly not be close whenever the length of the time interval is large compared with $\varepsilon^{-\kappa}$ (see [1]). Therefore, we expect that the optimal result is closeness of the streamfunctions on intervals of length $\mathrm{O}(\varepsilon^{-\kappa})$. Presumably, closeness will break down when going beyond that point. This is supported by the following *formal* argument. We write (4.1) in terms of ψ using $q = \Delta\psi + \beta y$. Transforming into moving coordinates, we obtain the equation

$$\frac{\partial}{\partial t}\Delta\Psi + \{\Psi, \Delta\Psi\} + \beta\Psi_\xi - c\Delta\Psi_\xi = \varepsilon\left(\Delta^2\Psi + F\right).$$

Formally, the linearization of this equation about a travelling wave Ψ is given by

$$L\tilde{\Psi} = \varepsilon\Delta\tilde{\Psi} + c\tilde{\Psi}_\xi - \Delta^{-1}(\{\tilde{\Psi}, \Delta\Psi\} + \{\Psi, \Delta\tilde{\Psi}\} + \beta\tilde{\Psi}_\xi).$$

For $\varepsilon > 0$, this operator generates an analytic semigroup on a suitable function space. If all points in its spectrum had a real part less than ε, the estimate

$$\|\mathrm{e}^{Lt}\| \leqslant M_\varepsilon \mathrm{e}^{\varepsilon t}$$

would hold. Assume that the constant M_ε can, in fact, be chosen independently of ε. Then the closeness of unperturbed and perturbed streamfunction could be concluded on intervals of length $\mathrm{O}(\varepsilon^{-\kappa})$ using the variation-of-constants formula.

We also remark that, even if the unperturbed and perturbed streamfunction stay close for all positive times, it might be necessary to use theorem 2 rather than [1, theorem 1], since the perturbed streamfunction may not be well defined for negative times.

5. An explicit Rossby wave

In this section, we compare the theoretical predictions of theorem 2 with numerical calculations for an exact solution to the viscous barotropic vorticity equation on the β-plane.

5.1. A model for meandering jets

We write equation (1.3) for the potential vorticity $q = \Delta\psi + \beta y$ in terms of the streamfunction ψ and obtain, with $f(x, y, t) = 0$,

$$\partial_t \Delta\psi + \{\psi, \Delta\psi + \beta y\} = \varepsilon\Delta^2\psi. \tag{5.1}$$

From [10, 12] an exact solution to the inviscid equation, $\varepsilon = 0$, is given by

$$\psi^0(x, y, t) = A\sin(k(x - ct))\sin(ly) \tag{5.2}$$

with the speed c satisfying

$$c = -\frac{\beta}{k^2 + l^2} < 0.$$

In a frame moving with the wave, $(\xi, \eta) = (x - ct, y)$, the velocity field is steady with streamfunction

$$\Psi^0(\xi, \eta) := \psi^0(\xi + ct, \eta, t) = A\sin(k\xi)\sin(l\eta).$$

For $\varepsilon > 0$ in equation (5.1), an exact solution is given by

$$\psi(x, y, t; \varepsilon) = Ae^{\gamma t}\sin(k(x - ct))\sin(ly). \tag{5.3}$$

The speed c is related to the wavenumbers k and l just as before, and the exponential decay rate γ satisfies

$$\gamma = -\varepsilon(k^2 + l^2) = \frac{\varepsilon\beta}{c}.$$

To investigate the dynamics of particles moving in the velocity fields satisfying (5.1), it is again useful to switch to a reference frame moving with the travelling wave, $(\xi, \eta) = (x - ct, y)$. In this moving frame, the streamfunction is given by

$$\Psi(\xi, \eta, t; \varepsilon) := \psi(\xi + ct, \eta, t; \varepsilon) = e^{\gamma t}\Psi^0(\xi, \eta) = Ae^{\gamma t}\sin(k\xi)\sin(l\eta)$$

and the Lagrangian dynamics are governed by

$$\begin{pmatrix}\dot\xi\\ \dot\eta\end{pmatrix} = J\nabla(\Psi(\xi, \eta, t; \varepsilon) + c\eta) = \begin{pmatrix}-e^{(\varepsilon\beta/c)t}\,Al\,\sin(k\xi)\cos(l\eta) - c\\ e^{(\varepsilon\beta/c)t}\,Ak\,\cos(k\xi)\sin(l\eta)\end{pmatrix}. \tag{5.4}$$

5.2. Theoretical predictions

First, consider the case $\varepsilon = 0$ where the velocity field is steady in the (ξ, η) reference frame:

$$\begin{pmatrix}\dot\xi\\ \dot\eta\end{pmatrix} = J\nabla(\Psi^0(\xi, \eta) + c\eta).$$

The contours of $\Psi^0(\xi, \eta)$ for this steady flow are shown in figure 1. The horizontal lines $\eta = 0$ and 1 are invariant under the flow for all $\varepsilon \geqslant 0$ and there are two distinct equilibrium points

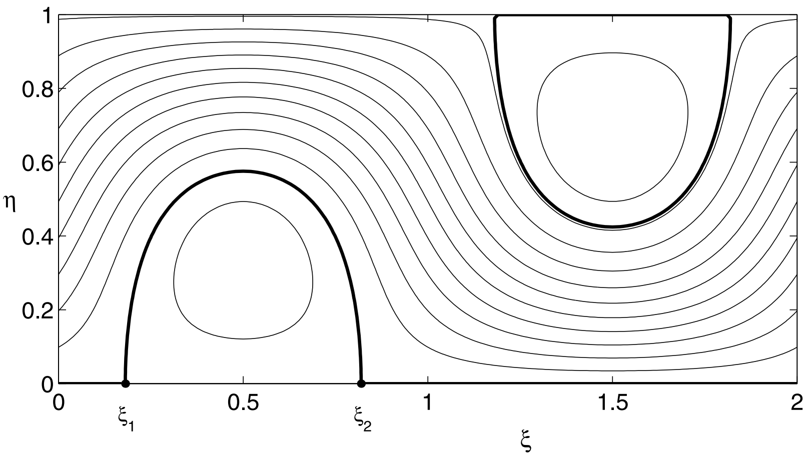


Figure 1. Streamlines for the steady solution to (5.1) where the parameters are chosen as in (5.7).

at $(\xi_1, 0)$ and $(\xi_2, 0)$ satisfying $Al\sin(k\xi_i) + c = 0$. With $-c, A, l > 0$, it follows that the equilibrium points at $\eta = 0$ exist provided

$$0 < -\frac{c}{Al} < 1.$$

The heteroclinic orbit $(\bar{\xi}, \bar{\eta})(t)$ connecting $(\xi_1, 0)$ with $(\xi_2, 0)$, together with the line $\xi = 0$, form a closed recirculation gyre where the trajectories are closed orbits in the (ξ, η)-phase space.

For $\varepsilon > 0$, equation (5.4) is given by

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = J\nabla(\Psi(\xi, \eta, t; \varepsilon) + c\eta) = J\nabla(\Psi^0(\xi, \eta) + c\eta) + (\mathrm{e}^{(\varepsilon\beta/c)\,t} - 1)J\nabla\Psi^0(\xi, \eta).$$

For every $\nu \in (\frac{1}{2}, 1)$, and with $\kappa = 1 - \nu < 1$, the ε-dependent perturbation clearly satisfies hypothesis 5 since

$$|\mathrm{e}^{(\varepsilon\beta/c)\,t} - 1| \leqslant C|\varepsilon t| \leqslant C\varepsilon^{\nu}$$

for $0 \leqslant t \leqslant 2\varepsilon^{-(1-\nu)}$ for some constant C that depends on β and c but not on t or ε.

Therefore, we can apply theorem 2 and conclude that the distance $d_{\text{theory}}(\tau, \varepsilon)$ between the perturbed stable and unstable manifolds of (5.4) near the heteroclinic orbit $(\bar{\xi}, \bar{\eta})(t)$ is given by

$$d_{\text{theory}}(\tau, \varepsilon) = \varepsilon(M_{\text{theory}} + \mathrm{O}(\varepsilon^{2\nu-1})) \tag{5.5}$$

where

$$\begin{aligned} M_{\text{theory}} &= \int_{-\infty}^{\infty} \Delta^2\Psi^0(\bar{\xi}(t), \bar{\eta}(t))\,\mathrm{d}t = -\frac{\beta^2}{c}\int_{-\infty}^{\infty} \bar{\eta}(t)\,\mathrm{d}t \\ &= -\frac{2\beta^2}{Akc}\int_0^{\eta_0} \frac{\eta}{\sqrt{\sin^2(l\eta) - (c\eta/A)^2}}\,\mathrm{d}\eta. \end{aligned} \tag{5.6}$$

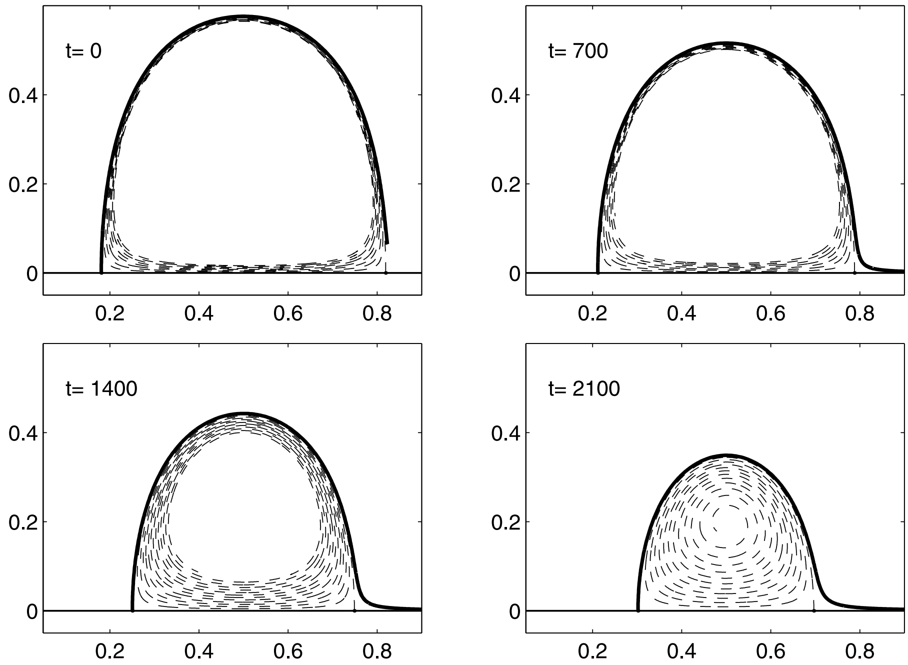


Figure 2. Unstable and stable manifolds, W_1^{u} and W_2^{s}, as computed for the exact solution to the barotropic potential vorticity equation in (5.1) with $\varepsilon = 1 \times 10^{-5}$, at times $t = 0, 700, 1400, 2100$. The full outer curve is W_1^{u} and the broken curve is W_2^{s}.

Here, η_0 is such that $A\sin(l\eta_0) + c\eta_0 = 0$. The second identity in (5.6) is a consequence of [1, lemma 7], while the last identity can be obtained upon exploiting the Hamiltonian nature of (5.4). The expansion (5.5) is valid on the time interval

$$(b|\ln\varepsilon|, \varepsilon^{-(1-\nu)}) = (b|\ln\varepsilon|, \varepsilon^{-\kappa})$$

where $b > 0$ is a certain constant. Since $c < 0$ and $\beta > 0$, we have $M_{\mathrm{theory}} > 0$, and it follows from (5.5) that $d_{\mathrm{theory}}(\tau, \varepsilon) > 0$ on the time interval given above.

5.3. Numerical simulations

For the numerical results to follow, the parameters are fixed at the values

$$\beta = 1.0 \qquad k = l = \pi \qquad A = 0.03. \tag{5.7}$$

This yields a wave speed of $c = -0.050\,66$ and the decay coefficient $\gamma = -19.74\,\varepsilon$. In dimensional values, selecting $\beta = 2.0 \times 10^{-11}\ \mathrm{m}^{-1}\ \mathrm{s}^{-1}$ (its value at about 30° latitude), and a length scale of 100 km, the parameters in (5.7) correspond to the eddy diffusivity parameter of the magnitude $\mathrm{O}(10^{-2})$ to $\mathrm{O}(10^{1})\ \mathrm{m}^2\ \mathrm{s}^{-1}$, i.e. to ε of the magnitude $\mathrm{O}(10^{-3})$ to $\mathrm{O}(10^{-6})$; see [13] for similar calculations.

For $\varepsilon > 0$, the plot in figure 1 still represents the instantaneous velocity field at the initial time $t = 0$. As t increases the locations of the zeros in the velocity field at $\eta = 0$ are determined by $\exp(\varepsilon\beta t/c)Al\sin(k\xi_i) + c = 0$. It follows that these zeros at $\xi_1(t)$ and $\xi_2(t)$ move towards one another and coalesce at $t = t_{\mathrm{c}} = \kappa/\varepsilon$, where $\kappa = c\ \ln(-c/Al)/\beta$. Using the parameter values from (5.7) yields $t_{\mathrm{c}} = 0.031\,45/\varepsilon$. For $t \geqslant t_{\mathrm{c}}$ there is no longer any notion

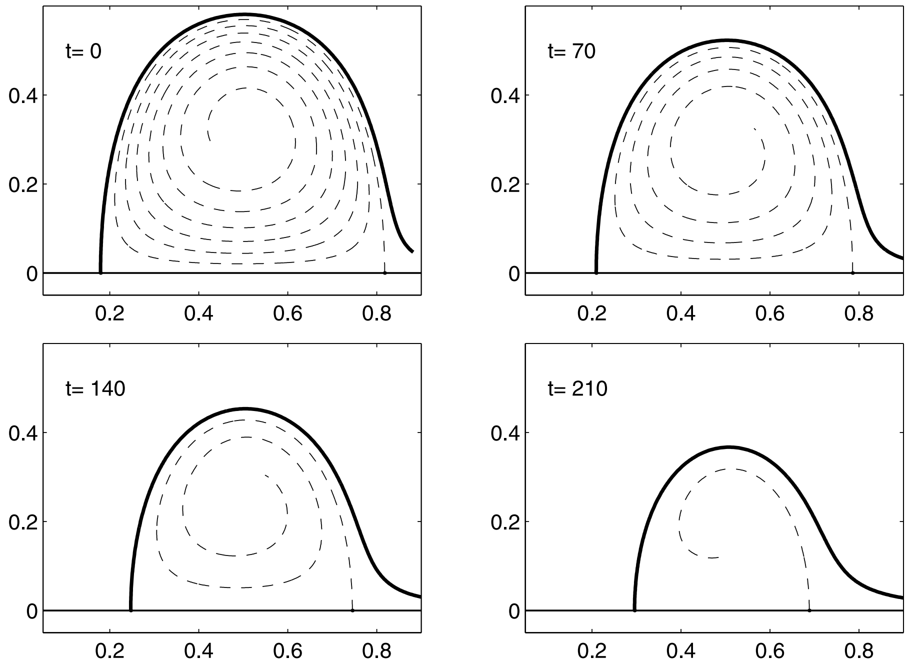
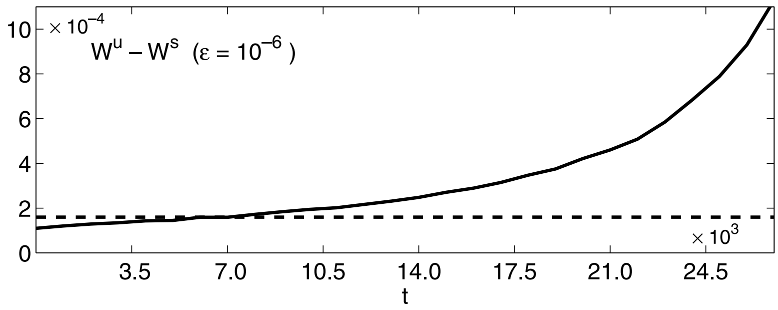


Figure 3. Computed manifolds, W_1^{u} and W_2^{s}, for $\varepsilon = 1 \times 10^{-4}$ at $t = 0, 70, 140, 210$.

of a recirculation gyre in the instantaneous velocity field. Note that this value of t is beyond the range where our theoretical predictions are valid. For convenience we write $\xi_1(t)$ and $\xi_2(t)$ to denote the time-dependent zeros of the velocity field (the coordinate $\eta = 0$ is implied). We have shown that, over finite time intervals of length $\varepsilon^{-\kappa}$ with $\kappa < 1$, there are hyperbolic trajectories $\gamma_1(t)$ and $\gamma_2(t)$ of (5.4) that are close to the zeros $(\xi_1(t), 0)$ and $(\xi_2(t), 0)$. The objective here is to approximate these trajectories and their associated manifolds numerically over a finite interval (see also [7]).

The parameter value $\varepsilon = 1 \times 10^{-4}$ is used to describe the procedures for approximating the stable and unstable manifolds of γ_1 and γ_2, and computing the transport out of the recirculation eddy. For this value of ε the zeros $\xi_1(t)$ and $\xi_2(t)$ exist up to the time $t_c = 314$. To approximate the unstable manifold of γ_1, denoted by W_1^{u}, a short line segment of initial conditions is evolved forward in time starting at $t_0 = -30$ and ending at $t = 300$. The initial line segment extends from $(\xi_1(-30), 0)$ to $(\xi_1(-30), 0.02)$. Similarly, the stable manifold at γ_2, denoted by W_2^{s}, is computed by evolving backward in time a segment of initial conditions starting at $t_{\mathrm{f}} = 310$ and ending at $t = 0$. This initial line segment extends from $(\xi_2(310), 0)$ to $(\xi_2(310), 0.02)$. Though the exact location of the distinguished trajectory is determined by our choice of initial data, the differences in the computed manifolds should be negligible for initial data chosen near the curves $\xi_1(t)$ and $\xi_2(t)$. The approximation of W_1^{u} and W_2^{s} is the same for the other parameter cases though the exact time scale depends on ε. Figures 2 and 3 include representative plots of the stable and unstable manifolds for $\varepsilon = 1 \times 10^{-5}$ and 1×10^{-4}.

The distance, $d_{\mathrm{num}}(\tau, \varepsilon)$, between the two manifolds is measured at $\xi = 0.5$, where the tangent lines to the two manifolds are horizontal. The transport or mass flux is approximated by the product $U(\tau)\, d_{\mathrm{num}}(\tau, \varepsilon)$, where U is the horizontal component of the velocity at the point midway between the two manifolds (at $\eta = 0.5$). Some details are summarized in table 1.



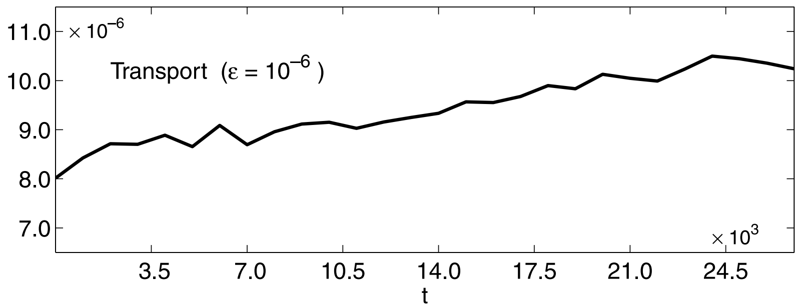


Figure 4. The top figure shows the computed distance between W_1^{u} and W_2^{s} for $\varepsilon = 1 \times 10^{-6}$. This distance is measured at the 'top' of the recirculation gyre at $\xi = 0.5$ (see figure 2, for example). The broken line indicates the theoretical value of the distance as computed using (5.5)–(5.7). The bottom figure shows the calculated mass transport out of the gyre. This transport is computed as the horizontal velocity at the midpoint of the gap multiplied by the length of the gap.

Table 1. The initialization times t_{f} for the stable manifolds, the distances between W^{u} and W^{s}, and the associated transport at time $\tau = 0$ are shown for various values of ε. The parameter values are $\beta = 1.0$, $k = l = \pi$ and $A = 0.03$.

ε	t_{f}	Distance at $\tau = 0$	Transport at $\tau = 0$
1×10^{-6}	62 000	1.10×10^{-4}	8.02×10^{-6}
5×10^{-6}	6 200	5.72×10^{-4}	4.16×10^{-5}
1×10^{-5}	3 100	1.14×10^{-3}	8.32×10^{-5}
5×10^{-5}	620	5.71×10^{-3}	4.16×10^{-4}
1×10^{-4}	310	1.14×10^{-2}	8.32×10^{-4}
2×10^{-4}	155	2.30×10^{-2}	1.67×10^{-3}
5×10^{-4}	62	6.00×10^{-2}	4.29×10^{-3}

Finally, we compare the theoretical prediction with the numerical simulations. As shown in the table above, the separation distance between the stable and unstable manifolds, measured at $\tau = 0$, and the phase space transport both scale linearly with ε, with

$$d_{\mathrm{num}}(\tau, \varepsilon) \approx 120\varepsilon.$$

On the other hand, evaluating the expression (5.6) for the Melnikov integral numerically, with parameters given by (5.7), we obtain

$$d_{\mathrm{theory}}(\tau, \varepsilon) \approx 160\varepsilon.$$

The difference can be explained as follows. As outlined above, we calculated the stable manifold by evolving backward in time a short line segment of initial conditions starting at

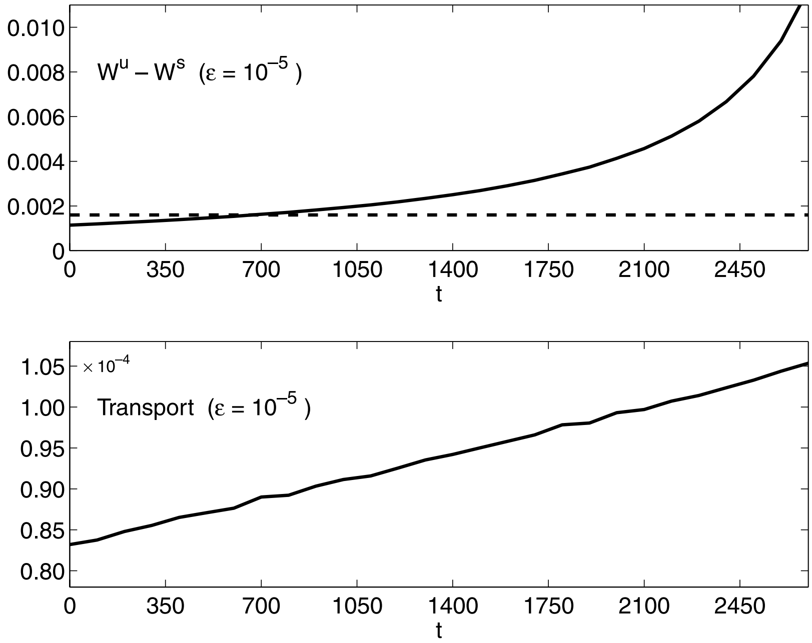


Figure 5. Distance between W_1^{u} and W_2^{s} and the calculated transport for $\varepsilon = 1 \times 10^{-5}$. The theoretical value of the distance is plotted as a broken line.

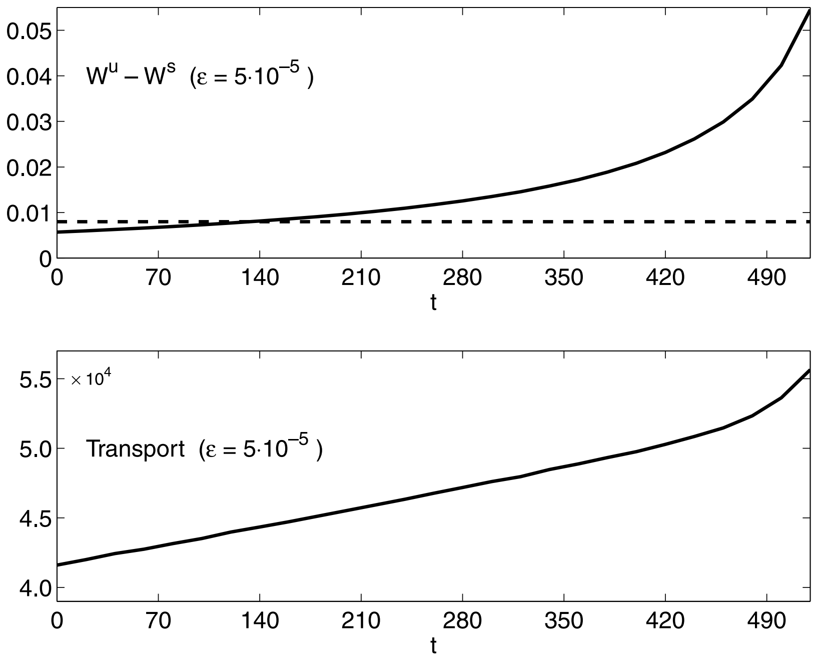


Figure 6. Distance between W_1^{u} and W_2^{s} and the calculated transport for $\varepsilon = 5 \times 10^{-5}$. The theoretical value of the distance is plotted as a broken line.

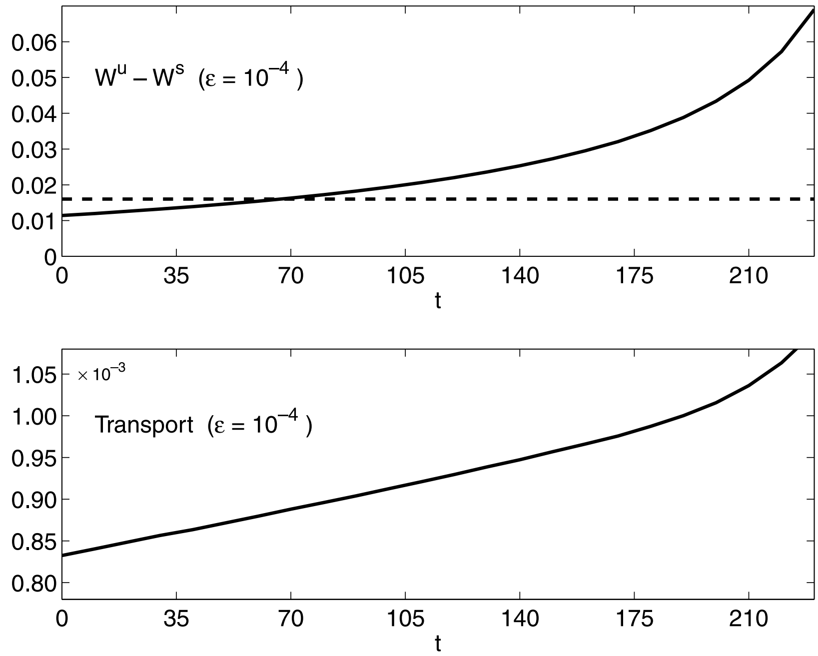


Figure 7. Distance between W_1^u and W_2^s and the calculated transport for $\varepsilon = 1 \times 10^{-4}$. The theoretical value of the distance is plotted as a broken line.

time t_f. The time t_f is, however, of the order of ε^{-1} and not of the order of $\varepsilon^{-1/2}$ (see the above table). Thus, the perturbation of the differential equation is large compared with $\varepsilon^{1/2}$ and our results do not necessarily apply at $\tau = 0$.

Figures 4–7 contain plots of the separation distance and transport as a function of t for the cases $\varepsilon = 1 \times 10^{-6}$, 1×10^{-5}, 5×10^{-5} and 1×10^{-4}, respectively. The theoretical value of the distance as given in (5.5) is plotted as a broken line. Note that it appears as if the plots in figures 4–7 scale linearly in ε. We believe that this is due to the special form of the non-autonomous perturbation in (5.4): the argument of the non-autonomous term is εt.

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