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Mixing, Transport and Coherent Structures

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Abstract. This is an excerpt from Mathematisches Forschungsinstitut Oberwolfach Report 4/2014, doi: 10.4171/OWR/2014/4, which contained extended abstracts from the Workshop on Mixing, Transport and Coherent Structures held at Oberwolfach in January, 2014. This extended abstract takes a broad and critical view on the existing methods on determining flow barriers in unsteady flows, while also calling for extensions to the ideas of stable and unstable manifolds to account for realistic flow situations.

Abstracts

Flow barriers in realistic flows, and their relationship to invariant manifolds

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Unambiguously identifying time-varying flow barriers in realistic unsteady flows faces many difficulties, since velocity fields are usually only available as spatially and temporally discrete data over a finite time. Despite a lack of agreement on the *definition* of flow barriers, a variety of methods for locating them are in usage. These include—but are not limited to—passively advecting tracers; Finite-Time Lyapunov Exponents; eigen/singular-vectors of transfer operators; complexity measures along trajectories; curves/surfaces of extremal stretching, or shearing, or flux, or attraction; ergodic quotient; and topological entropy. Interesting theoretical and computationally-efficient advances are being made at a rapid rate within the context of each proposed method of identifying flow barriers, as were explored by the many specialists attending this workshop.

Flow barrier detection: As more and more instances of these methods being applied to oceanographic, atmospheric and experimental data are being reported, it would be good to be certain of the legitimacy of the conclusions reached in such studies. As such, several broad questions are of concern:

1. In what way are these definitions related to each other? How is a curve of extremal deformation related to eigenfunctions of the transfer operator? Is a mesohyperbolic trajectory associated with an extremal flux? Questions of this sort can be posed between any two definitions, and demand our attention.

2. What is the accuracy of each definition in identifying flow barriers? Strangely, there is little analysis of this crucial question. While there is an obvious disincentive for a proponent of a particular definition to undertake such an investigation, this is surely scientifically necessary. A major problem in analysing this is having unequivocal unsteady flow barriers with which to compare.

3. And what is a flow barrier in an unsteady flow anyway? There are simple counter-examples which show that entities across which there is minimal (or zero) flux is a bad definition for this—but what is a flow barrier if it is not some entity associated with minimal transport across it? Should one of the proposed definitions— say a curve/surface of extremal stretching [10]—be used as *the* definition? This is unlikely to hold water, since competing definitions will surely vie for this, and more and more definitions are being developed thick and fast! Are there any features which are *essential* for a flow barrier to possess? For example:

4. Is frame invariance necessary? Not all methods for locating supposed flow barriers are frame invariant. A continuing debate on this issue is necessary. If, say, the boundary of an oceanic eddy is identified using some method based on observations from the earth's frame of reference, would the *same* boundary be identified from the reference frame of a ship travelling alongside it? Alternatively, if we want to quantify transport in our frame of reference, do we care what result we get in another frame of reference?

5. In what sense is time-periodicity implicit in some definitions? Real flows are patently *not* time-periodic, and thus recent methods that are being used and developed ostensibly avoid this restriction. However, consider the standard finite-time approach of defining a flow map P from a time t_1 to another time t_2. As an example, suppose the fact that all eigenvalues of a fixed point a of P are within the unit circle is used to argue the stability of a. This argument relies on $P^n(x) \to a$ as $n \to \infty$ for x near a, i.e., repeated applications of P. As a second example, suppose spectral—say, Koopman—methods based on Fourier series on $[t_1, t_2]$ are used. In both examples, the flow has been implicitly assumed time-periodic.

6. Is the time-variation captured? Flow barriers in unsteady flows must vary with time. If we think of t_1 and t_2 above as *fixed*, we are simply addressing the limited problem of just *one* iteration of an autonomous map P, thereby ignoring the time-dependence of flow barriers. For a definition/theory to be legitimate and useful, it must genuinely incorporate the time-varying nature; it should quite naturally be able to think of t_2 as a varying entity. Thus, addressing objects like P's invariant sets is surely meaningless in a time-varying setting.

Stable and unstable manifolds: Our initial ideas of flow barriers possibly arose from stable and unstable manifolds of stagnation points in steady flows, which indubitably form flow separators between regions of distinct fluid motion. The clear analogue in *unsteady* flows would be stable and unstable manifolds of hyperbolic trajectories/sets. The trouble is that to define these, one needs *infinite-time* flows. Thus, for an understanding of unsteady flow barriers and associated transport, further developments are called for:

a. Develop theory of finite-time invariant manifolds: There are some developments in this regard [12, 7, 8, 13, 9, 15], but theory to specifically locate finite-time flow barriers, and to establish connections with any diagnostic method, would be of tremendous value.

b. Quantify flux across intersection/non-intersecting invariant manifolds: While a segment of stable manifold might be considered a transport barrier, since such a segment by itself does not partition space, its location relative to other manifold segments controls how fluid is transported. In nonautonomous flows, this picture is changing with time, and moreover stable and unstable manifolds can intersect in an arbitrary fashion. How do such intersections—or lack of such—lead to transport? For time-periodic flows, transport across a flow barrier composed from such manifold segments can be quantified [14, 1]. For time-aperiodicity, a gate-surface idea [11] can be used to quantify time-dependent transport for nonautonomously *perturbed* 2D flows [2], but a general theory for more realistic flows is lacking. Since when we think of flow *barriers*, there is clearly an intuition regarding lack of transport across these, the development of theory to *quantify* transport explicitly across flow barriers in unsteady flows is highly relevant.

c. Test models with known invariant manifolds: There are unsteady nonchaotic models in 2D and 3D in which time-varying stable and unstable manifolds are explicitly known [4]; these apply also to finite time, and their construction explicitly addresses frame-dependence. Segments of invariant manifolds in nonautonomously perturbed chaotic 2D flows can also be computed to leading-order using recent theory [3]. In an alternative approach, recent work [5, 6] enables the determination of the control velocity needed to force segments of [un]stable manifolds and hyperbolic trajectories to almost follow user-specified time-aperiodic behaviour. Given the explicit nature of all of these models, they offer yet-to-be-exploited opportunities for testing accuracies of different methods of barrier detection.

In summary: Given the unequivocal nature of invariant manifolds as intuitive flow barriers in unsteady flows, their further development towards less idealised flows should be an important avenue we should pursue. These will further assist in the essential task of evaluating the accuracy of the multitudinous diagnostic methods in usage.

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