Nonautonomous control of stable and unstable manifolds in two-dimensional flows

Sanjeeva Balasuriya[a], Kathrin Padberg-Gehle[b]

[a]*School of Mathematical Sciences, University of Adelaide, Adelaide SA 5005, Australia*
[b]*Institute of Scientific Computing, Technische Universität Dresden, D-01062 Dresden, Germany*

Abstract

We outline a method for controlling the location of stable and unstable manifolds in the following sense. From a known location of the stable and unstable manifolds in a steady two-dimensional flow, the primary segments of the manifolds are to be moved to a user-specified time-varying location which is near the steady location. We determine the nonautonomous perturbation to the vector field required to achieve this control, and give a theoretical bound for the error in the manifolds resulting from applying this control. The efficacy of the control strategy is illustrated via a numerical example.

Keywords: controlling invariant manifolds, nonautonomous flow, flow barriers

1. Introduction

The role of stable and unstable manifolds in demarcating flow barriers in unsteady flows is well documented. Determining their location in a given unsteady flow regime is a problem which has attracted considerable attention, with many techniques continually being developed and refined in order to improve accuracy and efficiency [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

Viewing this problem from the reverse viewpoint leads to an intriguing question: is it possible to force stable and unstable manifolds to lie along *user-defined, time-varying locations*? The time-variation here is arbitrarily specified, and *not* confined to the popular time-periodic situation. If possible, this would yield an invaluable tool in controlling transport in micro- and nano-fluidic devices, with innumerable applications. This article answers this question in a specific setting: that of a nonautonomously perturbed two-dimensional system, in which the issue is to determine the nonautonomous perturbation which gives rise to the primary parts of the stable and unstable manifolds lying along prescribed one-dimensional curves at each instance in time. The theory is couched in terms of the perturbation being $\mathcal{O}(\varepsilon)$, and follows in spirit ideas of Melnikov theory [20, 21, 22, 23], particularly building on [24, 25, 12, 13]. The velocity

requirement for the prescribed manifolds to be achieved to leading-order in ε, and rigorous error bounds, are established.

The derived control strategy is tested on a time-aperiodic modification of the Taylor-Green flow [26, 27, 28, 24, 29]. Numerical diagnostics are compared with the prescribed stable manifolds, and excellent results are obtained.

While the method developed in this article is confined to perturbations of autonomous flows, it is to our knowledge the first theoretical contribution towards developing a control strategy for stable and unstable manifolds in nonautonomous flows. As such, it may serve as an important initial step towards building a more complete theory for the nonautonomous control of flow barriers.

2. Controlling stable manifold

Consider for $x \in \Omega$, a two-dimensional open connected set, the system

$$\dot{x} = f(x) \tag{1}$$

in which $f : \Omega \to \mathbb{R}^2$, and sufficient smoothness will be assumed (to be characterised shortly).

Hypothesis 2.1 (Saddle point at a). *The system (1) possesses a saddle fixed point a, that is, $f(a) = 0$ and $Df(a)$ possesses real eigenvalues λ_s and λ_u such that $\lambda_s < 0 < \lambda_u$.*

Then, a possesses corresponding one-dimensional stable and unstable manifolds. We will focus on *segments* of *one branch* of each of these manifolds, and denote them by Γ_s and Γ_u respectively. The segment of the stable manifold branch we will consider can be represented parametrically by

$$\Gamma_s := \{x_s(p) \ : \ p \in [S, \infty)\}$$

in which $x_s(t)$ is a solution to (1) with initial condition $x_s(0) \in \Gamma_s$, and $S \in (-\infty, 0]$ represents a finite backwards time until which the trajectory is evolved. Since $x_s(t) \to a$ as $t \to \infty$, $\bar{\Gamma}_s$ contains a, while the other end of the curve segment comprising Γ_s ends at the point $x_s(S)$. This definition precludes Γ_s from being a branch of a stable manifold which has infinite length, or a homo/hetero-clinic manifold associated with a fixed point. However, Γ_s could be any other finite length restriction of a branch of the stable manifold emanating from a, including a segment of any of the above two entities, or a segment of a manifold which has many rotations as it spirals out from a limit cycle. Similarly, let Γ_u be a segment of the unstable manifold of a, parametrisable as

$$\Gamma_u := \{x_u(p) \ : \ p \in (-\infty, U]\}$$

in which $x_u(t)$ is a solution to (1) with initial condition $x_u(0) \in \Gamma_u$, satisfying $x_u(t) \to a$ as $t \to -\infty$. Here $U \in [0, \infty)$ is a finite forward time until which the

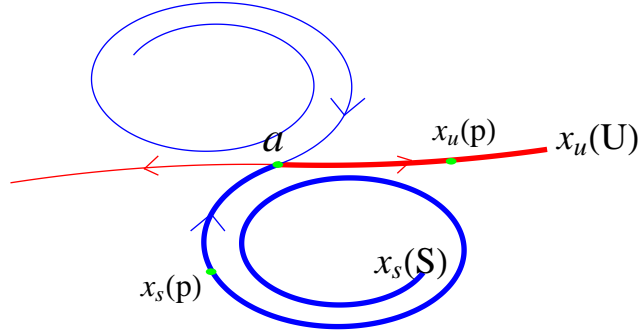Figure 1: Relevant segments of the stable manifold Γ_s (thick blue curve) and unstable manifold Γ_u (thick red curve) of the fixed point a, in the situation in which the stable manifold exhibits spiralling and the unstable manifold escapes to infinity.

trajectory is evolved. See Fig. 1 for an example of the finite segments Γ_s and Γ_u.

The goal is to determine a nonautonomous perturbation to the vector field in the form

$$\dot{x} = f(x) + \varepsilon g(x, t) \tag{2}$$

in which $\varepsilon \in [0, \varepsilon_0)$ where $\varepsilon_0 \ll 1$, such that Γ_s and Γ_u perturb to ε-close time-dependent entities which are *specified*. The following smoothness hypotheses on the functions f and g will be assumed, in which D represents the spatial (matrix) derivative operator in Ω.

Hypothesis 2.2 (Smoothness of f and g). *The functions $f : \Omega \to \mathbb{R}^2$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}^2$ satisfy:*

(f) $f \in \mathrm{C}^2(\Omega)$, and is such that there exists a constant C_f satisfying

$$\|f(x)\| + \|Df(x)\| + \|D^2 f(x)\| \leq C_f \quad \text{for all } x \in \Omega. \tag{3}$$

(g) $g \in \mathrm{C}^2(\Omega)$ for each $t \in \mathbb{R}$, and $g \in \mathrm{C}^1(\mathbb{R})$ for each $x \in \Omega$, and moreover there exists a constant C_g satisfying

$$\|g(x, t)\| + \|Dg(x, t)\| + \left\|\frac{\partial g}{\partial t}(x, t)\right\| \leq C_g \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}. \tag{4}$$

A note on the norms used in (3) and (4) is in order. The norm on $\mathbb{R}^2$ is the standard Euclidean norm. The norm on the 2×2 matrices Df and Dg is the

3

operator norm induced by the Euclidean norm. The norm on the $2 \times 2 \times 2$ entity $D^2 f$ is the induced operator norm associated with the above norms on vectors and matrices, i.e.,

$$\left\| \left(D^2 f \right) \right\| = \sup_{v \in \mathbb{R}^2 \setminus 0} \frac{\left\| \left(D^2 f \right) \, v \right\|}{\| v \|} \, , \tag{5}$$

in which the previously mentioned operator norm definition for 2×2 matrices is used in the numerator.

Since (2) is nonautonomous, it makes sense to view it in the augmented form

$$\left. \begin{array}{l} \dot{x} = f(x) + \varepsilon g(x, t) \\ \dot{t} = 1 \end{array} \right\} \tag{6}$$

with phase space now being $\Omega \times \mathbb{R}$. For (6) when $\varepsilon = 0$, the conditions stated for (1) provide for the presence of a hyperbolic trajectory (a, t) with two-dimensional stable and unstable manifolds.

From this point onwards, this Section will focus only on controlling the *stable* manifold, with the unstable manifold control description postponed to the subsequent Section. It will be necessary to restrict the stable manifold in time in the following sense. Let T_s be a finite time-value beyond which the restricted stable manifold is to be defined. The restricted two-dimensional stable manifold of (6) when $\varepsilon = 0$ will be represented in parametric form by

$$\Gamma_s := \{ (x_s(t - T_s + p), t) \; : \; (p, t) \in [S, \infty) \times [T_s, \infty) \} \, , \tag{7}$$

in which the notation Γ_s is retained with an abuse of notation. For each point $x_s(p)$ chosen in the time-slice $t = T_s$, $(x_s(t - T_s + p), t)$ represents the corresponding forward trajectory on Γ_s as it evolves with time t. Thus, the parameter p selects the trajectory. In the time-slice T_s, the restriction $p \geq S$ implies the relevant segment of the unperturbed stable manifold goes from $x_s(S)$ to a. As time t evolves for the unperturbed steady flow (1), $x_s(S + t - T_s)$ approaches the saddle fixed point a, and therefore the length of the restricted stable manifold in each time-slice gets shorter. This is illustrated in Fig. 2, in which trajectories associated with five p values are shown beginning with an "initial" point in the time-slice T_s. The "furthest" of these corresponds to the initial point $x_s(S)$ (i.e., $p = S$), and after three intermediate p values, the dashed trajectory is (a, t), which can be thought of as $p = \infty$ in (7). The stable manifold in the time-slice T_s is the curve segment (heavy magenta curve) which connects together all five starting points. Its time-evolution is indicated at two later time values in Fig. 2, also as heavy magenta curves.

Now, when $\varepsilon \neq 0$, and for any g satisfying the smoothness assumptions in Hypothesis 2.2, the hyperbolic trajectory (a, t) of (6) perturbs to an $\mathcal{O}(\varepsilon)$-close trajectory $(a_\varepsilon(t), t)$ which retains hyperbolicity. The proof of this is via exponential dichotomies [30, 31, 32], and as a consequence this trajectory retains stable and unstable manifolds which are ε-close to the original ones. In particular, it retains a stable manifold ε-close to (7). Here, we wish to find conditions of g
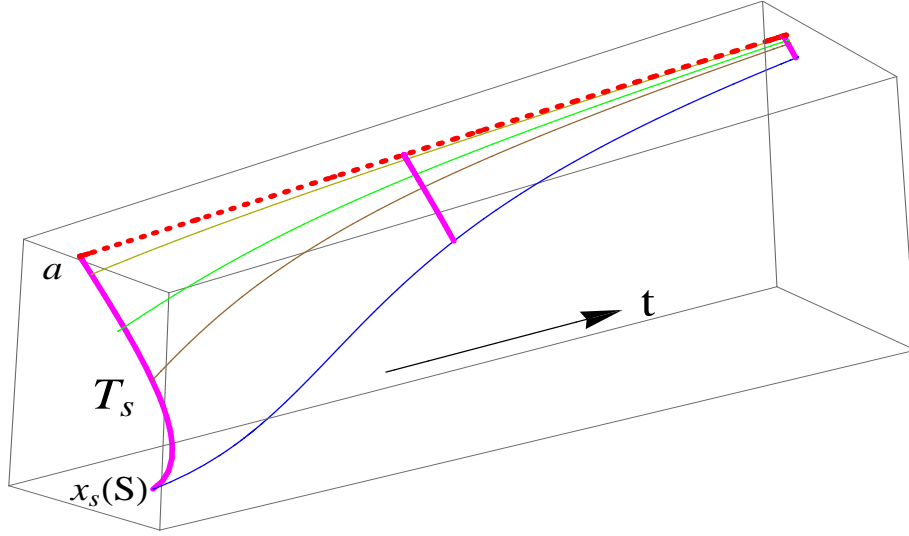
4

Figure 2: The evolution of the restricted stable manifold curve (heavy magenta curves), and the hyperbolic trajectory (dashed red line).

which result in the desired restricted stable manifold represented parametrically by

$$\Gamma_s^\varepsilon := \{(x_s^\varepsilon(p,t), t) \ : \ (p,t) \in [S,\infty) \times [T_s, \infty)\} \,, \tag{8}$$

where x_s^ε is assumed *given*, but satisfies several conditions to ensure consistency. To express these conditions, we first define

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \,, \tag{9}$$

the premultiplicative matrix which rotates vectors in $\mathbb{R}^2$ by $+\pi/2$.

Hypothesis 2.3 (Stable manifold requirements). *For each $t \geq T_s$, the quantity $\{x_s^\varepsilon(p,t) \ : \ p \geq S\}$ is a curve in Ω. These restricted stable manifold curves satisfy the following conditions.*

(a) *[Smoothness] There exists a constant $K_s > 0$ such that for all $(p,t) \in [S,\infty) \times [T_s, \infty)$ and for all $\varepsilon \in (0, \varepsilon_0)$,*

$$|x_s^\varepsilon(p,t)| + \left| \frac{\partial}{\partial t} x_s^\varepsilon(p,t) \right| + \left| \frac{\partial}{\partial p} x_s^\varepsilon(p,t) \right| + \left| \frac{\partial}{\partial \varepsilon} x_s^\varepsilon(p,t) \right| + \left| \frac{\partial^2}{\partial \varepsilon^2} x_s^\varepsilon(p,t) \right| + \left| \frac{\partial^3}{\partial \varepsilon^3} x_s^\varepsilon(p,t) \right| < K_s \,. \tag{10}$$

(b) *[Closeness] There exists a constant $C_s > 0$ such that for all $(p,t) \in [S,\infty) \times [T_s, \infty)$,*

$$|x_s^\varepsilon(p,t) - x_s(t - T_s + p)| + \left| \frac{\partial}{\partial p} \left(x_s^\varepsilon(p,t) - x_s(t - T_s + p) \right) \right| \leq C_s \, \varepsilon \,. \tag{11}$$
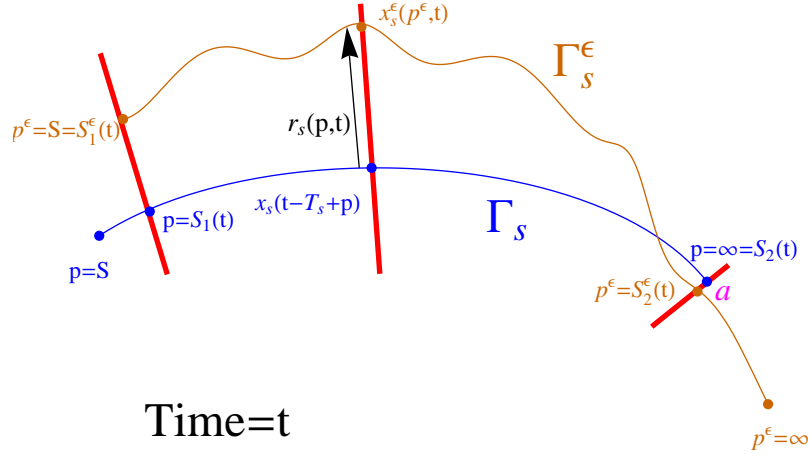
5

Figure 3: An illustration of the mappability condition (12) in the time-slice t. The heavy lines are in the normal direction Jf to Γ_s. The interval $[S_1(t), S_2(t)]$ is the p-interval for which the it is possible to map from Γ_s to Γ_s^ε by going in the normal direction Γ_s, while $\left[S_1^\varepsilon(t), S_2^\varepsilon(t)\right]$ is the corresponding interval for p^ε which parametrises Γ_s^ε. In this pictured situation, $S_2(t) = \infty$ and $S_1^\varepsilon(t) = S$.

(c) *[Limit] For each $t \geq T_s$, $\lim\limits_{p \to \infty} x_s^\varepsilon(p, t)$ is well defined.*

(d) *[Mappability] For each $t \geq T_s$, there exist intervals $[S_1(t), S_2(t))$ and $[S_1^\varepsilon(t), S_2^\varepsilon(t))$ —both of which are contained in $[S, \infty)$— and a scalar function $r_s(\bullet, t)$ defined on $[S_1(t), S_2(t))$ which satisfies*

$$x_s^\varepsilon(p^\varepsilon, t) = x_s(t - T_s + p) + r_s(p, t)\frac{Jf\left(x_s(t - T_s + p)\right)}{\left|Jf\left(x_s(t - T_s + p)\right)\right|}, \qquad (12)$$

such that the mapping $p \to p^\varepsilon$ from $[S_1(t), S_2(t))$ to $[S_1^\varepsilon(t), S_2^\varepsilon(t))$ defined through (12) is a diffeomorphism.

(e) *[Congruence at time T_s] For all $p \in [S_1(T_s), S_2(T_s))$,*

$$\left[f\left(x_s(p)\right)\right]^T \left[x_s^\varepsilon(p, T_s) - x_s(p)\right] = 0. \qquad (13)$$

Some of these hypotheses require explanation. Condition (b) states that for each fixed $t \in [T_s, \infty)$, the curves $x_s^\varepsilon(p, t)$ and $x_s(t - T_s + p)$ and their tangents in the t time-slice remain ε-close. Condition (c) requires that the end of the curve—that purportedly is on the hyperbolic trajectory $a_\varepsilon(t)$—is well-defined.

While becoming unbounded is already precluded by (a), (c) prevents $x_s^\varepsilon(p,t)$ behaving like, say, $\cos p$ for large p.

Condition (d) states that, in each time slice t, the restricted autonomous stable manifold segment and the required restricted nonautonomous stable manifold segment are mappable to one another by proceeding in the normal direction to each point $x_s(t-T_s-p)$, by a signed distance $r_s(p,t)$. The restriction of p and p^ε to these subintervals of $[S,\infty)$ is since some parts of the required Γ_s^ε may venture "beyond" the span of the normal direction to $\{x_s(t - T_s + p) \; ; \; p \in [S,\infty)\}$. This condition is illustrated by example in Fig. 3. The fact that the mapping from Γ_s to Γ_s^ε by going along the normal direction Jf from each point on Γ_s must be a diffeomorphism, prevents Γ_s^ε having self-intersections or twists which make the inverse function undefined.

Finally, the congruence condition (e) is elucidated in Fig. 4, which is the $t = T_s$ slice of phase space. For any fixed p, consider the point $x_s(p)$ on the unperturbed stable manifold, and suppose we draw a line perpendicular to $f(x_s(p))$. The congruence condition (13) at $t = T_s$ means that the p-parametrisation of $x_s^\varepsilon(p, T_s)$ is chosen such that $x_s^\varepsilon(p, T_s)$ lies exactly on this normal line drawn at $x_s(p)$. We have the freedom to do this for all mappable p in this one particular time-slice; this condition merely selects a choice of parametrisation of the one-dimensional curve obtained by intersecting Γ_s^ε with the time-slice $\{t = T_s\}$.

While the desired restricted stable manifold is given by (8), Hypotheses 2.3 further restricts the (p,t) values to lie in the set

$$\Xi_s := \{(p,t) \; : \; t \geq T_s \text{ and } S_1(t) \leq p < S_2(t)\} . \tag{14}$$

We will assume that the largest interval $[S_1(t), S_2(t))$ has been chosen for each t in order to fulfil the mappability condition of Hypothesis 2.3. For $(p,t,\varepsilon) \in \Xi_s \times (0,\varepsilon_0)$, we now define

$$M_s^\varepsilon(p,t) := [Jf(x_s(t - T_s + p))]^T \frac{x_s^\varepsilon(p,t) - x_s(t - T_s + p)}{\varepsilon} \tag{15}$$

and

$$B_s^\varepsilon(p,t) := [f(x_s(t - T_s + p))]^T \frac{x_s^\varepsilon(p,t) - x_s(t - T_s + p)}{\varepsilon} . \tag{16}$$

For a specified $x_s^\varepsilon(p,t)$, both M_s^ε and B_s^ε can be computed numerically based on the above expressions. Now, the required values of g (to leading-order) shall be expressed in terms of an orthogonal basis formed by projecting normally and tangentially to the autonomous stable manifold at $x_s(t-T_s+p)$ in the time-slice t.

Definition 2.1 (Control velocity for stable manifold). *The control velocity g satisfies the smoothness conditions of Hypothesis 2.2, and moreover is*
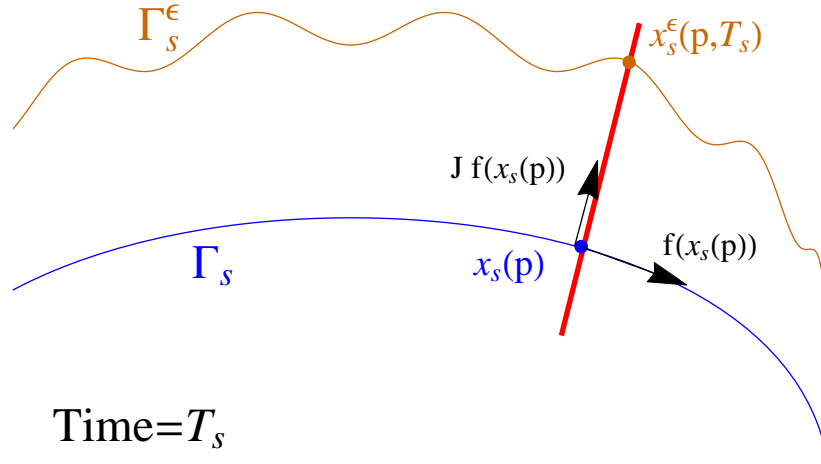
7

Figure 4: The congruence condition (13) in the time-slice $t = T_s$: at each p, the normal vector at $x_s(p)$ meets Γ_s^ε at the point $x_s^\varepsilon(p, T_s)$.

specified by

$$
\begin{aligned}
g^\perp \left(x_s(t - T_s + p), t \right) &:= \frac{\left[Jf\left(x_s(t - T_s + p) \right) \right]^T}{\left| f\left(x_s(t - T_s + p) \right) \right|} g\left(x_s(t - T_s + p), t \right) \\
&= \frac{\frac{\partial M_s^\varepsilon}{\partial t}(p, t) - \mathrm{Tr}\ (Df)\, M_s^\varepsilon(p, t)}{|f|}
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
g^\parallel \left(x_s(t - T_s + p), t \right) &:= \frac{\left[f\left(x_s(t - T_s + p) \right) \right]^T}{\left| f\left(x_s(t - T_s + p) \right) \right|} g\left(x_s(t - T_s + p), t \right) \\
&= \frac{|f|^2 \frac{\partial B_s^\varepsilon(p, t)}{\partial t} - f^T \left[(Df) + (Df)^T \right] \left[Jf M_s^\varepsilon(p, t) + f B_s^\varepsilon(p, t) \right]}{|f|^3}
\end{aligned}
\tag{18}
$$

in which f and Df in the above expressions are evaluated at $x_s(t - T_s + p)$,

By choosing g as above, it will be possible to achieve the desired nonautonomous stable manifold correct to $\mathcal{O}(\varepsilon)$. We will in Theorem 2.1 specify the error precisely. First, let us describe how to apply this control velocity computationally to achieve the desired stable manifold. Given the parametrised form $x_s^\varepsilon(p, t)$ of the restricted manifold Γ_s^ε, we proceed as follows.
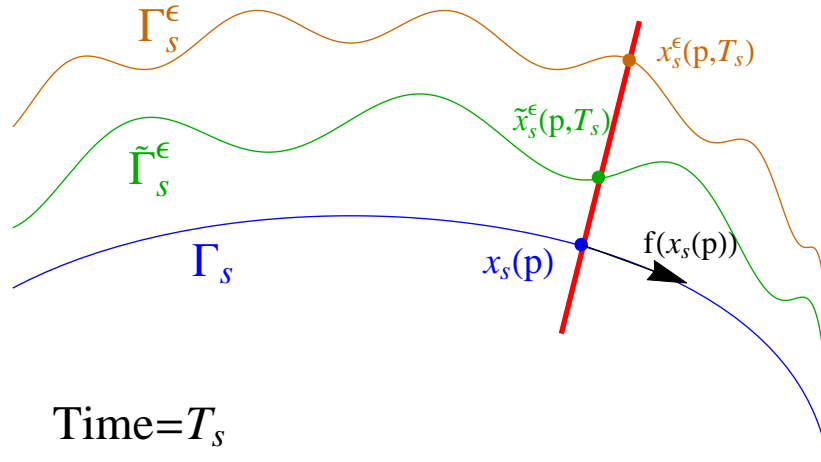
8

Figure 5: The congruence conditions (13) and (21) in the time-slice $t = 0$, illustrating that both the required (Γ_s^ε) and the real $(\tilde{\Gamma}_s^\varepsilon)$ stable manifolds have a congruent p-parametrisation at time zero.

1. Since full knowledge of the unperturbed steady flow (1) is presumed known, compute $x_s(p)$, and hence compute $f\left(x_s(t - T_s + p)\right)$, $Df\left(x_s(t - T_s + p)\right)$ and $\operatorname{Tr} Df\left(x_s(t - T_s + p)\right)$ as functions of (p, t);

2. Compute $M_s^\varepsilon(p, t)$ and $B_s^\varepsilon(p, t)$ from (15) and (16), recalling the restriction $(p, t) \in \Xi_s$;

3. Determine the t-derivatives of both $M_s^\varepsilon(p, t)$ and $B_s^\varepsilon(p, t)$, using a numerical method if needed;

4. Substitute these values into (17) and (18) to determine $g_s^\perp$ and $g_s^\parallel$, where in each time-slice t, the values are found along the restricted part of Γ_s lying between $x_s(t - T_s + S)$ and a;

5. Since $g^\perp$ and $g^\parallel$ give the components of g in the directions Jf and f respectively, this determines g at the locations $x_s(t - T_s + p)$ in time-slices t;

6. Extend g in any suitably relevant fashion to the spatial domain while being consistent with this requirement.

To evaluate this procedure, we need to compare the *desired* stable manifold with the *true* stable manifold resulting from applying the control velocity of Def. 2.1. We define this true stable manifold by

$$\tilde{\Gamma}_s^\varepsilon := \left\{ (\tilde{x}_s^\varepsilon(p, t), t) \ : \ (p, t) \in \Xi_s \right\} , \tag{19}$$

9

in which the $\tilde{x}_s^\varepsilon(p,t)$ is an exact trajectory of (2) in which g is as specified in Def. 2.1. For each p, the trajectory $\tilde{x}_s^\varepsilon(p,t)$ lies on the associated true perturbed manifold $\tilde{\Gamma}_s^\varepsilon$, with the t parametrising the time evolution. Thus, $|\tilde{x}_s^\varepsilon(p,t) - a_\varepsilon(t)| \to 0$ as $t \to \infty$ for any p. Moreover, the parametrisation p can be chosen so that $\tilde{x}_s^\varepsilon(p,t)$ is $\mathcal{O}(\varepsilon)$-close to $x_s(t - T_s + p)$, that is, there exists a constant $\tilde{C}_s$ such that

$$|\tilde{x}_s^\varepsilon(p,t) - x_s(t - T_s + p)| + \left| \frac{\partial}{\partial p} \left(\tilde{x}_s^\varepsilon(p,t) - x_s(t - T_s + p) \right) \right| \leq \varepsilon \tilde{C}_s \qquad (20)$$

for $(p, t, \varepsilon) \in \Xi_s \times [0, \varepsilon_0)$. The expectation is that $\tilde{C}_s \approx C_s$ as given in (11), since the purported Γ_s^ε as given in (8) and parametrised by $x_s^\varepsilon(p,t)$ will be forced to be close to the true restricted stable manifold $\tilde{\Gamma}_s^\varepsilon$ which is parametrised by $\tilde{x}_s^\varepsilon(p,t)$. We note from Fig. 5 that it is possible to choose the parametrisation p on $\tilde{x}_s^\varepsilon(p,t)$ such that it too lies exactly on the normal vector drawn at $x_s(p)$ in the time-slice $t = T_s$. That is, analogous to the congruence condition (13) for the desired stable manifold, we require

$$[f(x_s(p))]^T [\tilde{x}_s^\varepsilon(p, T_s) - x_s(p)] = 0 \qquad (21)$$

for the *true* stable manifold. (For more details about characterising such tangential movement of perturbed manifolds, see [12].) Now, we write

$$\tilde{x}_s^\varepsilon(p,t) = x_s^\varepsilon(p,t) + e_s(p,t), \qquad (22)$$

in which the $e_s(p,t)$s represent the error in the restricted stable manifold at time t and parameter p. An illustration of $e_s(p,t)$ is provided in Fig. 6. While in the time-slice $t = T_s$ the parameter p was chosen congruently as shown in Fig. 5, the t-evolution of $\tilde{x}_s^\varepsilon(p,t)$ generated by the flow (2), and the specified t-evolution of $x_s^\varepsilon(p,t)$, mean that this congruence is *not* preserved for either the true or the required stable manifold at general t. Thus $e_s(p,t)$ has in general both a normal and a tangential term, for which bounds are provided below.

Theorem 2.1 (Error in stable manifold). *Assume the control velocity g satisfies Def. 2.1, and define*

$$e_s^\perp(p,t) := \frac{Jf(x_s(t - T_s + p))}{|f(x_s(t - T_s + p))|} e_s(p,t) \text{ and } e_s^\parallel(p,t) := \frac{f(x_s(t - T_s + p))}{|f(x_s(t - T_s + p))|} e_s(p,t). \qquad (23)$$

The normal component is bounded by

$$\left| e_s^\perp(p,t) \right| \leq \left[C_s C_g + \frac{C_s^2 C_f}{2} \right] \varepsilon^2 \frac{\int_t^\infty |f(x_s(\tau - T_s + p))| \exp\left[\int_\tau^t \mathrm{Tr}\left[Df(x_s(\xi - T_s + p)) \right] \mathrm{d}\xi \right] \mathrm{d}\tau}{|f(x_s(t - T_s + p))|}, \qquad (24)$$

for $(p, t, \varepsilon) \in \Xi_s \times (0, \varepsilon_0)$, and satisfies the limits

$$\lim_{t \to \infty} \left| e_s^\perp(p,t) \right| \leq -\frac{\left[C_s C_g + \frac{C_s^2 C_f}{2} \right] \varepsilon^2}{\lambda_s}, \quad \lim_{p \to \infty} \left| e_s^\perp(p,t) \right| \leq \frac{\left[C_s C_g + \frac{C_s^2 C_f}{2} \right] \varepsilon^2}{\lambda_u}, \qquad (25)$$
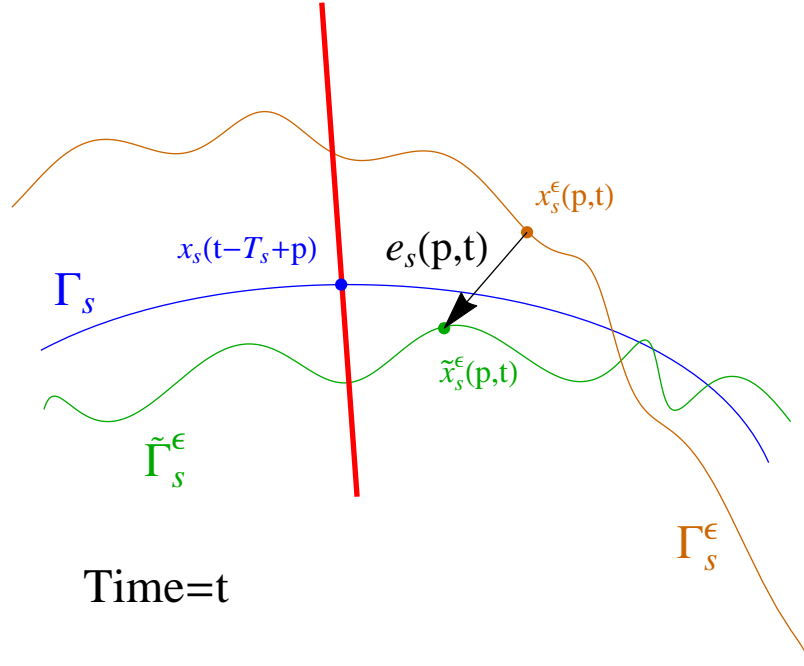
10

Figure 6: The intersections of the unperturbed (Γ_s), required (Γ_s^ε) and true ($\tilde{\Gamma}_s^\varepsilon$) restricted stable manifolds in a general time-slice t.

as long as these limits can be taken within the domain Ξ_s. The tangential component of the error is bounded by

$$\left| e_s^\parallel(p,t) \right| \leq \varepsilon^2 \left(C_s C_g + \frac{C_s^2 C_f}{2} \right) \left| f\left(x_s(t-T_s+p)\right) \right|$$

$$\times \int_{T_s}^t \frac{\left| f\left(x_s(\tau-T_s+p)\right) \right| + 2C_f \int_\tau^\infty \left| f\left(x_s(\zeta-T_s+p)\right) \right| \exp\left[\int_\zeta^\tau \mathrm{Tr}\left[Df\left(x_s(\xi-T_s+p)\right) \right] \mathrm{d}\xi \right] \mathrm{d}\zeta}{\left| f\left(x_s(\tau-T_s+p)\right) \right|^2} \mathrm{d}\tau$$

$$(26)$$

for $(p,t,\varepsilon) \in \Xi_s \times (0,\varepsilon_0)$, and (subject to being in Ξ_s) obeys the limiting behaviour

$$\lim_{t\to\infty} \left| e_s^\parallel(p,t) \right| = 0 \,, \quad \lim_{p\to\infty} \left| e_s^\parallel(p,t) \right| \leq \varepsilon^2 \frac{\left(2C_g C_s + C_s^2 C_f \right)\left(\lambda_u + 2C_f \right)}{-2\lambda_s \lambda_u} \left[1 - e^{\lambda_s(t-T_s)} \right] \,. \quad (27)$$

Proof: See Section 4. □

   Theorem 2.1 provides a precise statement on why e_s is $\mathcal{O}(\varepsilon^2)$ for $(p,t,\varepsilon) \in \Xi_s \times (0,\varepsilon_0)$. The improper integral in (24) is convergent, since as shown in the

proof the integrand exhibits exponential decay. Consequently, so is the interior integral in (26). The limiting behaviour in (25) indicates how the perpendicular component of the restricted stable manifold error remains bounded in the limits as time goes to infinity, or in each time-slice as the foot of the manifold (i.e., the hyperbolic trajectory $a_\varepsilon(t)$) is approached. The fact that the tangential component of the error approaches zero as $t \to \infty$ is a consequence of the *restricted* nature of the stable manifold. In this limit, the length of the restricted stable manifold collapses to zero. Put another way, this occurs because both x_s^ε and $\tilde{x}_s^\varepsilon$ undergo exponentially contracting behaviour in the form $e^{\lambda_s t}$ in the tangential direction.

The detailed derivation of all this result is given in Section 4, with a numerical example demonstrating the accuracy of the control strategy given in Section 6.

3. Controlling unstable manifold

We next focus on determining the control velocity g in controlling the unstable manifold to have user-specified behaviour. The results are analogous to those of the stable manifold but require careful statement since there is no requirement for the unstable manifold to have any relationship to the stable one.

Let T_u be a finite time-value before which the restricted unstable manifold is to be quantified. We represent the restricted two-dimensional unstable manifold of (6) when $\varepsilon = 0$ by

$$\Gamma_u := \{(x_u(t - T_u + p), t) \ : \ (p, t) \in (-\infty, U] \times (-\infty, T_u]\} , \qquad (28)$$

in which $x_u(\textbf{.})$ is the trajectory lying along the unstable manifold. The restricted unstable manifold which we desire to achieve in the $\varepsilon \neq 0$ system will be represented by

$$\Gamma_u^\varepsilon := \{(x_u^\varepsilon(p, t), t) \ : \ (p, t) \in (-\infty, U] \times (-\infty, T_u]\} , \qquad (29)$$

for which we impose the conditions:

Hypothesis 3.1 (Unstable manifold requirements). *For each $t \leq T_u$, the quantity $\{x_u^\varepsilon(p, t) \ : \ p \leq U\}$ is a curve in Ω. These restricted unstable manifold curves satisfy the following conditions.*

(a) *[Smoothness] There exists a constant $K_u > 0$ such that for all $(p, t) \in (-\infty, U] \times (-\infty, T_u]$, and all $\varepsilon \in (0, \varepsilon_0)$,*

$$|x_u^\varepsilon(p,t)| + \left|\frac{\partial}{\partial t}x_u^\varepsilon(p,t)\right| + \left|\frac{\partial}{\partial p}x_u^\varepsilon(p,t)\right| + \left|\frac{\partial}{\partial \varepsilon}x_u^\varepsilon(p,t)\right| + \left|\frac{\partial^2}{\partial \varepsilon^2}x_u^\varepsilon(p,t)\right| + \left|\frac{\partial^3}{\partial \varepsilon^3}x_u^\varepsilon(p,t)\right| < K_u .$$

$$(30)$$

12

(b) *[Closeness] There exists a constant $C_u > 0$ such that for all $(p,t) \in (-\infty, U] \times (-\infty, T_u]$,*

$$|x_u^\varepsilon(p,t) - x_u(t - T_u + p)| + \left| \frac{\partial}{\partial p} \left(x_u^\varepsilon(p,t) - x_u(t - T_u + p) \right) \right| \le C_u\, \varepsilon \,. \tag{31}$$

(c) *[Limit] For each $t \le T_u$, $\lim\limits_{p \to -\infty} x_u^\varepsilon(p,t)$ is well defined.*

(d) *[Mappability] For each $t \le T_u$, there exist intervals $(U_1(t), U_2(t)]$ and $(U_1^\varepsilon(t), U_2^\varepsilon(t)]$ —both of which are contained in $(\infty, U]$— and a scalar function $r_u(\textbf{.}, t)$ defined on $(U_1(t), U_2(t)]$ which satisfies*

$$x_u^\varepsilon(p^\varepsilon, t) = x_u(t - T_u + p) + r_u(p,t) \frac{Jf\left(x_u(t - T_u + p)\right)}{|Jf\left(x_u(t - T_u + p)\right)|} \,, \tag{32}$$

such that the mapping $p \to p^\varepsilon$ from $(U_1(t), U_2(t)]$ to $(U_1^\varepsilon(t), U_2^\varepsilon(t)]$ defined through (32) is a diffeomorphism.

(e) *[Congruence at time T_u] For all $p \in (U_1(T_u), U_2(T_u)]$,*

$$[f\left(x_u(p)\right)]^T [x_u^\varepsilon(p, T_u) - x_u(p)] = 0 \,. \tag{33}$$

The set of (p,t) for which control is to be achieved is restricted to the set

$$\Xi_u = \{(p,t) \;:\; t \le T_u \text{ and } U_1(t) < p \le U_2(t)\} \,, \tag{34}$$

where the largest interval $(U_1(t), U_2(t)]$ is chosen for each t in order to fulfil the mappability condition of Hypothesis 3.1. Now, for a prescribed restricted unstable manifold $x_u^\varepsilon(p,t)$ we define the functions

$$M_u^\varepsilon(p,t) := [Jf\left(x_u(t - T_u + p)\right)]^T \frac{x_u^\varepsilon(p,t) - x_u(t - T_u + p)}{\varepsilon} \tag{35}$$

and

$$B_u^\varepsilon(p,t) := [f\left(x_u(t - T_u + p)\right)]^T \frac{x_u^\varepsilon(p,t) - x_u(t - T_u + p)}{\varepsilon} \,, \tag{36}$$

valid for $(p,t,\varepsilon) \in \Xi_u \times (0, \varepsilon_0)$.

Definition 3.1 (Control velocity for unstable manifold). *The control velocity g satisfies the smoothness conditions of Hypothesis 2.2, and moreover is specified in normal and tangential components on the original unstable manifold by*

$$
\begin{aligned}
g^\perp\left(x_u(t - T_u + p), t\right) \quad &:= \quad \frac{[Jf\left(x_u(t - T_u + p)\right)]^T}{|f\left(x_u(t - T_u + p)\right)|} g\left(x_u(t - T_u + p), t\right) \\
&= \quad \frac{\frac{\partial M_u^\varepsilon}{\partial t}(p,t) - \text{Tr}\,(Df)\, M_u^\varepsilon(p,t)}{|f|}
\end{aligned} \tag{37}
$$

13

and

$$
\begin{aligned}
g^{\parallel}\left(x_u(t - T_u + p), t\right) &:= \frac{\left[f\left(x_u(t - T_u + p)\right)\right]^T}{\left|f\left(x_u(t - T_u + p)\right)\right|} g\left(x_u(t - T_u + p), t\right) \\
&= \frac{|f|^2 \frac{\partial B_u^\varepsilon(p,t)}{\partial t} - f^T\left[(Df) + (Df)^T\right]\left[JfM_u^\varepsilon(p,t) + fB_u^\varepsilon(p,t)\right]}{|f|^3}
\end{aligned}
$$
(38)

in which f and Df in the above expressions are evaluated at $x_u(t - T_u + p)$.

To characterise the resulting error, we define the true unstable manifold by

$$
\tilde{\Gamma}_u^\varepsilon := \left\{ (\tilde{x}_u^\varepsilon(p,t), t) \ : \ (p,t) \in \Xi_u \right\},
$$
(39)

rather than (29), in which $\tilde{x}_u^\varepsilon(p,t)$ is an exact trajectory of (2) which lies on the associated true perturbed manifold $\tilde{\Gamma}_u^\varepsilon$. Analogous to the congruence condition (33), we require

$$
\left[f\left(x_u(p)\right)\right]^T \left[\tilde{x}_u^\varepsilon(p, T_u) - x_u(p)\right] = 0
$$
(40)

for the true unstable manifold, and define the error $e_u(p,t)$ through

$$
\tilde{x}_u^\varepsilon(p,t) = x_u^\varepsilon(p,t) + e_u(p,t).
$$
(41)

Theorem 3.1 (Error in unstable manifold). *Assume the control velocity g satisfies Def. 3.1, and define*

$$
e_u^\perp(p,t) := \frac{Jf\left(x_u(t - T_u + p)\right)}{\left|f\left(x_u(t - T_u + p)\right)\right|} e_u(p,t) \ \text{and} \ e_u^{\parallel}(p,t) := \frac{f\left(x_u(t - T_u + p)\right)}{\left|f\left(x_u(t - T_u + p)\right)\right|} e_u(p,t).
$$
(42)

The normal component is bounded by

$$
\left|e_u^\perp(p,t)\right| \le \left[C_u C_g + \frac{C_u^2 C_f}{2}\right]\varepsilon^2 \frac{\int_{-\infty}^t \left|f\left(x_u(\tau - T_u + p)\right)\right| \exp\left[\int_\tau^t \mathrm{Tr}\left[Df\left(x_u(\xi - T_u + p)\right)\right]\mathrm{d}\xi\right]\mathrm{d}\tau}{\left|f\left(x_u(t - T_u + p)\right)\right|},
$$
(43)

for $(p,t,\varepsilon) \in \Xi_u \times (0, \varepsilon_0)$, and satisfies

$$
\lim_{t \to -\infty} \left|e_u^\perp(p,t)\right| \le \frac{\left[C_u C_g + \frac{C_u^2 C_f}{2}\right]\varepsilon^2}{\lambda_u} \ , \ \lim_{p \to -\infty} \left|e_u^\perp(p,t)\right| \le -\frac{\left[C_u C_g + \frac{C_u^2 C_f}{2}\right]\varepsilon^2}{\lambda_s},
$$
(44)

as long as these limits can be taken within the domain Ξ_u. The tangential component of the error is bounded by

$$
\begin{aligned}
\left|e_u^{\parallel}(p,t)\right| \ \le \ & \varepsilon^2 \left(C_u C_g + \frac{C_u^2 C_f}{2}\right) \left|f\left(x_u(t - T_u + p)\right)\right| \\
& \times \int_t^{T_u} \frac{\left|f\left(x_u(\tau - T_u + p)\right)\right| + 2C_f \int_{-\infty}^\tau \left|f\left(x_u(\zeta - T_u + p)\right)\right| \exp\left[\int_\zeta^\tau \mathrm{Tr}\left[Df\left(x_u(\xi - T_u + p)\right)\right]\mathrm{d}\xi\right]\mathrm{d}\zeta}{\left|f\left(x_u(\tau - T_u + p)\right)\right|^2}\mathrm{d}\tau
\end{aligned}
$$
(45)

for $(p, t, \varepsilon) \in \Xi_u \times (0, \varepsilon_0)$, and (subject to being in Ξ_u) obeys the limiting behaviour

$$\lim_{t \to -\infty} \left| e_u^{\parallel}(p, t) \right| = 0 \,, \quad \lim_{p \to -\infty} \left| e_u^{\parallel}(p, t) \right| \leq \varepsilon^2 \frac{\left(2C_g C_u + C_u^2 C_f \right) \left(-\lambda_s + 2C_f \right)}{-2\lambda_s \lambda_u} \left[1 - e^{\lambda_u (t - T_u)} \right] \,. \tag{46}$$

Proof: See Section 5. $\square$

4. Proof of Theorem 2.1

We begin by introducing the notation

$$y(t) := x_s \left(t - T_s + p \right) \,. \tag{47}$$

Now, we argue that $e_s(p, t)$ is bounded for $(p, t, \varepsilon) \in \Xi_s \times (0, \varepsilon_0)$. This is since

$$
\begin{aligned}
|e_s(p, t)| &= |\tilde{x}_s^{\varepsilon}(p, t) - x_s^{\varepsilon}(p, t)| \\
&= |\tilde{x}_s^{\varepsilon}(p, t) - a_{\varepsilon}(t) + a_{\varepsilon}(t) - a + a - y(t) + y(t) - x_s^{\varepsilon}(p, t)| \\
&\leq |\tilde{x}_s^{\varepsilon}(p, t) - a_{\varepsilon}(t)| + |a_{\varepsilon}(t) - a| + |a - y(t)| + |y(t) - x_s^{\varepsilon}(p, t)|
\end{aligned}
$$

The first term goes to zero as $t \to \infty$, since $\tilde{x}_s^{\varepsilon}(p, t)$ is an exact solution to the perturbed equation (2) which lies on the stable manifold of $a_{\varepsilon}(t)$. The p selects a particular trajectory on this stable manifold, and thus this limit holds for any $p \geq S$. Similarly, since $y(t) = x_s(t - T_s + p)$ is on the stable manifold of a, the third term also goes to zero as $t \to \infty$. Thus, these two terms are bounded. The term $|a_{\varepsilon}(t) - a| \leq \varepsilon C$ for some constant C for $t \in [T_s, \infty)$ since the hyperbolic trajectory remains $\mathcal{O}(\varepsilon)$-close to the unperturbed one [30, 31]. Finally, the term $|y(t) - x_s^{\varepsilon}(p, t)| \leq C_s \varepsilon$ by Hypothesis 2.3. Therefore, $|e_s(p, t)|$ is bounded.

In contrast to $M_s^{\varepsilon}(p, t)$ in (15), we define on Ξ_s an "M_s^{ε} with error" function

$$
\begin{aligned}
\hat{M}_s^{\varepsilon}(p, t) &:= \left[Jf \left(x_s(t - T_s + p) \right) \right]^T \frac{\tilde{x}_s^{\varepsilon}(p, t) - x_s(t - T_s + p)}{\varepsilon} \\
&= \left[Jf \left(y(t) \right) \right]^T \frac{\left[x_s^{\varepsilon}(p, t) + e_s(p, t) \right] - y(t)}{\varepsilon} \,. \tag{48}
\end{aligned}
$$

Since the smoothness assumptions on f and g (Hypothesis 2.2) ensure $\tilde{x}_s^{\varepsilon}(p, t)$ is differentiable, we differentiate $\hat{M}_s^{\varepsilon}$ with respect to t leading to

$$
\begin{aligned}
\varepsilon \frac{\partial \hat{M}_s^{\varepsilon}}{\partial t}(p, t) &= \left[Jf \left(y(t) \right) \right]^T \left[\frac{\partial \left[x_s^{\varepsilon}(p, t) + e_s(p, t) \right]}{\partial t} - \frac{\partial y(t)}{\partial t} \right] + \left[J \frac{\partial f \left(y(t) \right)}{\partial t} \right]^T \left[x_s^{\varepsilon}(p, t) + e_s(p, t) - y(t) \right] \\
&= \left[Jf \left(y(t) \right) \right]^T \left[f \left(x_s^{\varepsilon}(p, t) + e_s(p, t) \right) + \varepsilon g \left(x_s^{\varepsilon}(p, t) + e_s(p, t), t \right) - f \left(y(t) \right) \right] \\
&\quad + \left[J \, Df \left(y(t) \right) \frac{\partial y(t)}{\partial t} \right]^T \left[x_s^{\varepsilon}(p, t) - y(t) + e_s(p, t) \right] \\
&= \left[Jf \left(y(t) \right) \right]^T \left[f \left(x_s^{\varepsilon}(p, t) + e_s(p, t) \right) - f \left(y(t) \right) \right] + \varepsilon \left[Jf \left(y(t) \right) \right]^T g \left(x_s^{\varepsilon}(p, t) + e_s(p, t), t \right) \\
&\quad + \left[J \, Df \left(y(t) \right) f \left(y(t) \right) \right]^T \left[x_s^{\varepsilon}(p, t) - y(t) \right] + \left[J \, Df \left(y(t) \right) f \left(y(t) \right) \right]^T e_s(p, t) \tag{49}
\end{aligned}
$$

15

In the above calculations, the facts that $x_s^\varepsilon(p,t) + e_s(p,t)$ is an exact solution to the nonautonomous equation (2), and $y(t) = x_s(t - T_s + p)$ similarly satisfies the autonomous equation (1), have been used. We note from Taylor's theorem that

$$
\begin{aligned}
f\left(x_s^\varepsilon(p,t) + e_s(p,t)\right) &= f\left(y(t)\right) + Df\left(y(t)\right)\left(x_s^\varepsilon(p,t) + e_s(p,t) - y(t)\right) \\
&+ \frac{1}{2}\left(x_s^\varepsilon(p,t) + e_s(p,t) - y(t)\right)^T D^2 f\left(\xi_1\right)\left(x_s^\varepsilon(p,t) + e_s(p,t) - y(t)\right) \quad (50)
\end{aligned}
$$

and that

$$
g\left(x_s^\varepsilon(p,t) + e_s(p,t), t\right) = g\left(y(t), t\right) + Dg\left(\xi_2, t\right)\left(x_s^\varepsilon(p,t) + e_s(p,t) - y(t)\right) \quad (51)
$$

for some points $\xi_{1,2} \in \Omega$. Substituting these expansions into (49) and dividing by ε, we arrive at

$$
\begin{aligned}
\frac{\partial \hat{M}_s^\varepsilon}{\partial t}(p,t) &= \left[Jf\left(y(t)\right)\right]^T g\left(y(t), t\right) + \left[Jf\left(y(t)\right)\right]^T Df\left(y(t)\right)\frac{x_s^\varepsilon(p,t) + e_s(p,t) - y(t)}{\varepsilon} \\
&+ \left[J\,Df\left(y(t)\right)f\left(y(t)\right)\right]^T \frac{x_s^\varepsilon(p,t) - y(t)}{\varepsilon} + \left[J\,Df\left(y(t)\right)f\left(y(t)\right)\right]^T \frac{e_s(p,t)}{\varepsilon} \\
&+ \left[Jf\left(y(t)\right)\right]^T E_s(p,t).
\end{aligned}
$$

Here $E_s(p,t)$ is a higher-order term satisfying

$$
|E_s(p,t)| \le \varepsilon\left[\tilde{C}_s C_g + \frac{\tilde{C}_s^2 C_f}{2}\right], \quad (52)
$$

using (20) and the bounds in Hypotheses 2.2 and 2.3, valid for $(p, t, \varepsilon) \in \Xi_s \times (0, \varepsilon_0)$. Using the easily verifiable identity $\left[Jb\right]^T A + \left[JAb\right]^T = \left(\mathrm{Tr}\, A\right)\left[Jb\right]^T$ for 2×1 vectors b and 2×2 matrices A, we get $\left[Jf\right]^T (Df) + \left[J(Df)f\right]^T = \mathrm{Tr}\,(Df)\left[Jf\right]^T$, and hence

$$
\begin{aligned}
\frac{\partial \hat{M}_s^\varepsilon}{\partial t}(p,t) &= \left[Jf\left(y(t)\right)\right]^T g\left(y(t), t\right) + \mathrm{Tr}\left[Df\left(y(t)\right)\right]\left[Jf\left(y(t)\right)\right]^T \frac{x_s^\varepsilon(p,t) - y(t)}{\varepsilon} \\
&+ \mathrm{Tr}\left[Df\left(y(t)\right)\right]\left[Jf\left(y(t)\right)\right]^T \frac{e_s(p,t)}{\varepsilon} + \left[Jf\left(y(t)\right)\right]^T E_s(p,t).
\end{aligned}
$$

Now, comparing the definitions of $M_s^\varepsilon(p,t)$ and $\hat{M}_s^\varepsilon(p,t)$, the above can be written as

$$
\begin{aligned}
\frac{\partial M_s^\varepsilon}{\partial t}(p,t) &+ \frac{1}{\varepsilon}\frac{\partial}{\partial t}\left\{\left[Jf\left(y(t)\right)\right]^T e_s(p,t)\right\} = \left[Jf\left(y(t)\right)\right]^T E_s(p,t) + \left[Jf\left(y(t)\right)\right]^T g\left(y(t), t\right) \\
&+ \mathrm{Tr}\left[Df\left(y(t)\right)\right] M_s^\varepsilon(p,t) + \mathrm{Tr}\left[Df\left(y(t)\right)\right]\left[Jf\left(y(t)\right)\right]^T \frac{e_s(p,t)}{\varepsilon}. \quad (53)
\end{aligned}
$$

We would like e_s to be $\mathcal{O}(\varepsilon^2)$, which is yet to be established. So we choose our purported $\mathcal{O}(\varepsilon^0)$ terms to be zero by setting

$$
\frac{\partial M_s^\varepsilon}{\partial t}(p,t) = \left[Jf\left(y(t)\right)\right]^T g\left(y(t), t\right) + \mathrm{Tr}\left[Df\left(y(t)\right)\right] M_s^\varepsilon(p,t).
$$

Under this condition, we note that

$$g^{\perp}\left(y(t),t\right) := \frac{\left[Jf\left(y(t)\right)\right]^T}{\left|f\left(y(t)\right)\right|} g\left(y(t),t\right) = \frac{\frac{\partial M_s^{\varepsilon}}{\partial t}(p,t) - \text{Tr}\left[Df\left(y(t)\right)\right] M_s^{\varepsilon}(p,t)}{\left|f\left(y(t)\right)\right|},$$

which is exactly the control strategy defined in (17). Thus, the remaining terms in (53) are

$$\frac{\partial}{\partial t}\left\{\left[Jf\left(y(t)\right)\right]^T e_s(p,t)\right\} - \text{Tr}\left[Df\left(y(t)\right)\right]\left[Jf\left(y(t)\right)\right]^T e_s(p,t) = \varepsilon\left[Jf\left(y(t)\right)\right]^T E_s(p,t).$$

We multiply through by the integrating factor

$$\mu(p,t) := \exp\left[\int_t^{T_s} \text{Tr}\left[Df\left(y(\xi)\right)\right] \mathrm{d}\xi\right], \qquad (54)$$

giving the expression

$$\frac{\partial}{\partial t}\left\{\mu(p,t)\left[Jf\left(y(t)\right)\right]^T e_s(p,t)\right\} = \varepsilon\mu(p,t)\left[Jf\left(y(t)\right)\right]^T E_s(p,t)$$

which we integrate from a general t value to a large value L to obtain

$$\mu(p,L)\left[Jf\left(y(L)\right)\right]^T e_s(p,L) - \mu(p,t)\left[Jf\left(y(t)\right)\right]^T e_s(p,t) = \varepsilon\int_t^L \mu(p,\tau)\left[Jf\left(y(\tau)\right)\right]^T E_s(p,\tau)\mathrm{d}\tau. \qquad (55)$$

Before applying $L \to \infty$ in (55), we need to argue that this limit is defined. Now

$$
\begin{aligned}
\left|\mu(p,L)\left[Jf\left(y(L)\right)\right]^T\right| &= e^{\int_L^{T_s} \text{Tr}[Df(y(\xi))]\mathrm{d}\xi}\left|f\left(y(L)\right)\right| \\
&\to e^{\int_L^{T_s}(\lambda_s+\lambda_u)\mathrm{d}\xi} K e^{\lambda_s(L-T_s+p)} \\
&= K e^{-(\lambda_u+\lambda_s)(L-T_s)} e^{\lambda_s(L-T_s+p)} \\
&= K e^{\lambda_s p} e^{-\lambda_u(L-T_s)}
\end{aligned}
$$

where we have used the facts that $\text{Tr}\,(Df)$ approaches the sum of the eigenvalues at a as its argument approaches a, and that $|f|$ has exponential decay with rate λ_s as its argument approaches a along the stable manifold. Here, K is some constant, and since $p \geq S$ and $\lambda_s < 0$, the first exponential term is bounded by $e^{\lambda_s S}$. Thus, the quantity $\left|\mu(p,L)\left[Jf\left(y(L)\right)\right]^T\right|$ decays exponentially in L with rate $-\lambda_u$ as $L \to \infty$. Since $e_s(p,t)$ is bounded, when taking the limit $L \to \infty$ in (55), the first term on the left-hand side disappears. On the other hand, the boundedness of $E_s(p,t)$ given in (52) in conjunction with the fact that the other terms inside the integrand have $e^{-\lambda_u \tau}$ behaviour (by the same argument used above) implies that the improper integral on the right converges. Thus we get

$$-\mu(p,t)\left[Jf\left(y(t)\right)\right]^T e_s(p,t) = \varepsilon\int_t^{\infty} \mu(p,\tau)\left[Jf\left(y(\tau)\right)\right]^T E_s(p,\tau)\mathrm{d}\tau. \qquad (56)$$

Now we note from (52) that

$$-\varepsilon\left[\tilde{C}_s C_g + \frac{\tilde{C}_s^2 C_f}{2}\right]|f\left(y(\tau)\right)| \le [Jf\left(y(\tau)\right)]^T E_s(p,t) \le \varepsilon\left[\tilde{C}_s C_g + \frac{\tilde{C}_s^2 C_f}{2}\right]|f\left(y(\tau)\right)|\,.$$

Dividing (56) by $\mu(p,t)\,|f\left(y(\tau)\right)|$ and utilising the above bounds, we get

$$\left|e_s^\perp(p,t)\right| \le \left[\tilde{C}_s C_g + \frac{\tilde{C}_s^2 C_f}{2}\right]\varepsilon^2 \frac{\int_t^\infty |f\left(y(\tau)\right)|\,e^{\int_\tau^t \mathrm{Tr}[Df(y(\xi))]\mathrm{d}\xi}\mathrm{d}\tau}{|f\left(y(t)\right)|}\,,$$

which is a genuine bound since the integrand of the improper integral exhibits exponential decay, and hence the integral is bounded. Now, from (11) and (20) we know that $\left|\tilde{C}_s - C_s\right| \to 0$ as $\varepsilon \to 0$, and hence for sufficiently small ε_0 we can replace $\tilde{C}_s$ above with C_s, which leads directly to (24). Moreover, the value of $\left|e_s^\perp(p,t)\right|$ is bounded as $t \to \infty$, which is seen by a L'Hôpital's rule application to the above:

$$\begin{aligned}
\lim_{t\to\infty}\left|e_s^\perp(p,t)\right| &\le& \left[C_s C_g + \frac{C_s^2 C_f}{2}\right]\varepsilon^2 \lim_{t\to\infty}\frac{-\,|f\left(y(t)\right)|}{\frac{\partial}{\partial t}\,|f\left(y(t)\right)|} \\
&=& -\left[C_s C_g + \frac{C_s^2 C_f}{2}\right]\varepsilon^2 \lim_{t\to\infty}\frac{1}{\frac{\partial}{\partial t}\ln|f\left(y(t)\right)|}\,.
\end{aligned}$$

But since $|f\left(y(t)\right)| \sim e^{\lambda_s(t-T_s+p)}$, the limit above is $1/\lambda_s$, and we obtain the result in (25). The limit $p \to \infty$ at each fixed t is easiest computed with the formal replacements $|f\left(y(t)\right)| \sim e^{\lambda_s(t-T_s+p)}$ and $\mathrm{Tr}\,Df\left(y(\xi)\right) \sim \lambda_u + \lambda_s$. Thus,

$$\begin{aligned}
\lim_{p\to\infty}\left|e_s^\perp(p,t)\right| &\le& \left[C_s C_g + \frac{C_s^2 C_f}{2}\right]\varepsilon^2 \int_t^\infty \frac{e^{\lambda_s(\tau-T_s+p)}}{e^{\lambda_s(t-T_s+p)}}\exp\left[\int_\tau^t (\lambda_s + \lambda_u)\,\mathrm{d}\xi\right]\mathrm{d}\tau \\
&=& \left[C_s C_g + \frac{C_s^2 C_f}{2}\right]\varepsilon^2 e^{\lambda_u t}\int_t^\infty e^{-\lambda_u \tau}\,\mathrm{d}\tau = \left[C_s C_g + \frac{C_s^2 C_f}{2}\right]\varepsilon^2 \frac{1}{\lambda_u}\,.
\end{aligned}$$

Hence $\left|e^\perp(p,t)\right|$ exhibits the limiting behaviour in (25).

To evaluate the velocity requirement in the direction tangential to the manifold, we proceed analogously and define

$$\begin{aligned}
\hat{B}_s^\varepsilon(p,t) &:=& [f\left(x_s(t-T_s+p)\right)]^T\,\frac{\tilde{x}_s^\varepsilon(p,t) - x_s(t-T_s+p)}{\varepsilon} \\
&=& [f(y(t))]^T\frac{[x_s^\varepsilon(p,t) + e_s(p,t)] - y(t)}{\varepsilon}\,. \quad (57)
\end{aligned}$$

Taking the t-derivative of $\hat{B}_s^\varepsilon$ leads to

$$
\begin{aligned}
\varepsilon \frac{\partial \hat{B}_s^\varepsilon}{\partial t}(p,t) &= [f(y(t))]^T \left[\frac{\partial}{\partial t}[x_s^\varepsilon(p,t) + e_s(p,t)] - \frac{\partial y(t)}{\partial t} \right] + \left[\frac{\partial f(y(t))}{\partial t} \right]^T [x_s^\varepsilon(p,t) + e_s(p,t) - y(t)] \\
&= [f(y(t))]^T [f(x_s^\varepsilon(p,t) + e_s(p,t)) + \varepsilon g(x_s^\varepsilon(p,t) + e_s(p,t),t) - f(y(t))] \\
&\quad + \left[Df(y(t)) \frac{\partial y(t)}{\partial t} \right]^T [x_s^\varepsilon(p,t) + e_s(p,t) - y(t)] \\
&= \varepsilon [f(y(t))]^T g(x_s^\varepsilon(p,t) + e_s(p,t),t) + [f(y(t))]^T [f(x_s^\varepsilon(p,t) + e_s(p,t)) - f(y(t))] \\
&\quad + [Df(y(t)) f(y(t))]^T [x_s^\varepsilon(p,t) + e_s(p,t) - y(t)] .
\end{aligned}
$$

Applying the expansions (50) and (51) and dividing by ε gives

$$
\begin{aligned}
\frac{\partial \hat{B}_s^\varepsilon}{\partial t}(p,t) &= [f(y(t))]^T g(y(t),t) + [f(y(t))]^T E_s(p,t) \\
&\quad + \left\{ [f(y(t))]^T Df(y(t)) + [Df(y(t)) f(y(t))]^T \right\} \frac{x_s^\varepsilon(p,t) + e_s(p,t) - y(t)}{\varepsilon}
\end{aligned}
$$

in which $E_s(p,t)$ satisfies (52). Therefore,

$$
\begin{aligned}
\frac{\partial B_s^\varepsilon}{\partial t}(p,t) + \frac{1}{\varepsilon} \frac{\partial}{\partial t} \left\{ [f(y(t))]^T e_s(p,t) \right\} &= [f(y(t))]^T E_s(p,t) + [f(y(t))]^T g(y(t),t) \\
+ f^T \left[Df + (Df)^T \right]\Big|_{y(t)} \frac{x_s^\varepsilon(p,t) - y(t)}{\varepsilon} &+ f^T \left[Df + (Df)^T \right]\Big|_{y(t)} \frac{e_s(p,t)}{\varepsilon}.
\end{aligned} \tag{58}
$$

Now, using (15) and (16), we write

$$
\frac{x_s^\varepsilon(p,t) - y(t)}{\varepsilon} = \frac{Jf(y(t))}{|f(y(t))|^2} M_s^\varepsilon(p,t) + \frac{f(y(t))}{|f(y(t))|^2} B_s^\varepsilon(p,t), \tag{59}
$$

since $M_s^\varepsilon / |f|$ and $B_s^\varepsilon / |f|$ are the projections of the vector on the left-hand side of (59) into the orthogonal directions given by $Jf/|f|$ and $f/|f|$ respectively. Substituting into (58) yields

$$
\begin{aligned}
\frac{\partial B_s^\varepsilon}{\partial t}(p,t) + \frac{1}{\varepsilon} \frac{\partial}{\partial t} \left\{ [f(y(t))]^T e_s(p,t) \right\} &= [f(y(t))]^T E_s(p,t) + [f(y(t))]^T g(y(t),t) \\
&\quad + \frac{f^T \left[Df + (Df)^T \right] Jf}{|f|^2}\Bigg|_{y(t)} M_s^\varepsilon(p,t) + \frac{f^T \left[Df + (Df)^T \right] f}{|f|^2}\Bigg|_{y(t)} B_s^\varepsilon(p,t) \\
&\quad + f^T \left[Df + (Df)^T \right]\Big|_{y(t)} \frac{e_s(p,t)}{\varepsilon}.
\end{aligned} \tag{60}
$$

We select the terms we plan to be $\mathcal{O}(\varepsilon^0)$ above to be zero, giving

$$
\begin{aligned}
[f(y(t))]^T g(y(t),t) &= \frac{\partial B_s^\varepsilon}{\partial t}(p,t) - \frac{f^T \left[Df + (Df)^T \right] Jf}{|f|^2}\Bigg|_{y(t)} M_s^\varepsilon(p,t) \\
&\quad - \frac{f^T \left[Df + (Df)^T \right] f}{|f|^2}\Bigg|_{y(t)} B_s^\varepsilon(p,t)
\end{aligned}
$$

19

Thus

$$
\begin{aligned}
g_s^{\parallel}(y(t),t) \quad &:= \quad \frac{[f(y(t))]^T}{|f(y(t))|} g(y(t),t) \\
&= \quad \frac{1}{|f(y(t))|} \frac{\partial B_s^{\varepsilon}}{\partial t}(p,t) - \left. \frac{f^T\left[Df + (Df)^T\right]}{|f|^3}\right|_{y(t)} (JfM_s^{\varepsilon}(p,t) + fB_s^{\varepsilon}(p,t)),
\end{aligned}
$$

which is the tangential component of the control velocity required, as given in equation (18). Under this choice, the remaining terms in (60) yield

$$
\frac{\partial}{\partial t}\left\{[f(y(t))]^T e_s(p,t)\right\} = \varepsilon[f(y(t))]^T E_s(p,t) + f^T\left[Df + (Df)^T\right]\Big|_{y(t)} e_s(p,t). \quad (61)
$$

Expressing $e_s(p,t)$ in terms of the orthogonal unit vectors $Jf/|f|$ and $f/|f|$ as

$$
e_s(p,t) = \left\{[Jf]^T(y(t))\, e_s(p,t)\right\}\frac{Jf(y(t))}{|f(y(t))|^2} + \left\{f^T(y(t))\, e_s(p,t)\right\}\frac{f(y(t))}{|f(y(t))|^2}
$$

enables (61) to be written as

$$
\frac{\partial}{\partial t}\left[f^T e_s(p,t)\right] = \varepsilon f^T E_s(p,t) + \frac{f^T\left[Df + (Df)^T\right]Jf}{|f|^2}(Jf)^T e_s + \frac{f^T\left[Df + (Df)^T\right]f}{|f|^2}f^T e_s
$$

where the argument $y(t)$ in each of the f terms has been suppressed for convenience. Thus we have the equation

$$
\frac{\partial}{\partial t}\left[f^T e_s(p,t)\right] - \frac{f^T\left[Df + (Df)^T\right]f}{|f|^2}\left[f^T e_s(p,t)\right] = \varepsilon f^T E_s(p,t) + \frac{f^T\left[Df + (Df)^T\right]Jf}{|f|}\frac{(Jf)^T}{|f|}e_s. \quad (62)
$$

The left-hand side can be simplified with the observation

$$
\frac{\partial}{\partial t}[f(y(t))] = Df(y(t))\frac{\partial}{\partial t}[y(t)] = Df(y(t))f(y(t)), \quad (63)
$$

whose transpose is

$$
\frac{\partial}{\partial t}\left[f^T(y(t))\right] = f^T(y(t))[Df]^T(y(t)). \quad (64)
$$

Therefore, we note that

$$
\frac{\partial}{\partial t}\left[f^T(y(t))f(y(t))\right] = f^T[Dff] + \left[f^T(Df)^T\right]f = f^T\left[Df + (Df)^T\right]f
$$

Hence,

$$
\frac{f^T\left[Df + (Df)^T\right]f}{|f|^2} = \frac{1}{|f|^2}\frac{\partial}{\partial t}\left[f^T f\right] = \frac{1}{|f|^2}\frac{\partial}{\partial t}|f|^2 = \frac{\partial}{\partial t}\ln|f|^2,
$$

and therefore (62) can be written as

$$
\frac{\partial}{\partial t}\left[f^T e_s(p,t)\right] - \frac{\partial}{\partial t}\left[\ln|f|^2\right]\left[f^T e_s(p,t)\right] = \varepsilon f^T E_s(p,t) + \frac{f^T\left[Df + (Df)^T\right]Jf}{|f|}e_s^\perp\,,
$$

(65)

where we have used the fact that $e_s^\perp = (Jf)^T e_s/|f|$. Multiplying (65) through by the integrating factor $|f(y(t))|^{-2}$, and integrating from T_s to a general t value yields

$$
\frac{f^T(y(t))}{|f(y(t))|^2}e_s(p,t) = \int_{T_s}^t \frac{\varepsilon f^T E_s(p,\tau) + \left.\frac{f^T\left[Df+(Df)^T\right]Jf}{|f|}\right|_{y(\tau)}e_s^\perp}{|f(y(\tau))|^2}\mathrm{d}\tau\,.
$$

(66)

In obtaining (66), the congruence conditions (13) and (21) as indicated in Fig. 5 have been used to get eliminate the boundary term at $t = T_s$. Now, we bound the integrand in (66) using (24), (52) and Hypothesis 2.2, and with the understanding that $\tilde{C}_s$ can be replaced with C_s for suitably small ε_0:

$$
\left|\frac{\varepsilon f^T E_s(p,\tau) + \frac{f^T\left[Df+(Df)^T\right]Jf}{|f|}e_s^\perp}{|f|^2}\right|
$$

$$
\leq \frac{\left|f^T\right|}{|f|^2}\left|\varepsilon|E_s| + \left|Df + (Df)^T\right|\left|e_s^\perp\right|\right|
$$

$$
\leq |f|^{-1}\left|\varepsilon^2\left(C_s C_g + \frac{C_s^2 C_f}{2}\right) + 2C_f\left|e_s^\perp\right|\right|
$$

$$
\leq |f|^{-1}\varepsilon^2\left(C_s C_g + \frac{C_s^2 C_f}{2}\right)\left[1 + 2C_f\frac{\int_\tau^\infty |f(y(\zeta))|\,e^{\int_\zeta^\tau \mathrm{Tr}[Df(y(\xi))]\mathrm{d}\xi}\mathrm{d}\zeta}{|f(y(\tau))|}\right]
$$

$$
=: |f|^{-1}\varepsilon^2\left(C_s C_g + \frac{C_s^2 C_f}{2}\right)H(p,\tau)
$$

which defines H as the term in the square brackets, and we note that H is bounded in τ since the τ-dependent quotient in H has a finite limit as established in (25). Therefore from (66),

$$
\begin{aligned}
\left|e_s^\parallel(p,t)\right| &= \left|\frac{f^T(y(t))}{|f(y(t))|}e_s(p,t)\right| \\
&\leq \varepsilon^2\left(C_g C_s + \frac{C_s^2 C_f}{2}\right)|f(y(t))|\left|\int_{T_s}^t \frac{H(p,\tau)}{|f(y(\tau))|}\mathrm{d}\tau\right| \\
&= \varepsilon^2\left(C_g C_s + \frac{C_s^2 C_f}{2}\right)|f(y(t))|\int_{T_s}^t \frac{|f(y(\tau))| + 2C_f\int_\tau^\infty |f(y(\zeta))|\,e^{\int_\zeta^\tau \mathrm{Tr}[Df(y(\xi))]\mathrm{d}\xi}\mathrm{d}\zeta}{|f(y(\tau))|^2}\mathrm{d}\tau\,.
\end{aligned}
$$

Writing in the ∞/∞ form, L'Hôpital's Rule can be used to show that the above goes to zero as $t \to \infty$:

$$
\lim_{t\to\infty}\frac{\int_{T_s}^t \frac{H(p,\tau)}{|f(y(\tau))|}\mathrm{d}\tau}{|f(y(t))|^{-1}} = \lim_{t\to\infty}\frac{\frac{H(p,t)}{|f(y(t))|}}{-|f(y(t))|^{-2}} = -\lim_{t\to\infty}H(p,t)|f(y(t))| = 0\,,
$$

since H is bounded and $|f(y(t))| \to |f(a)| = 0$. To take the $p \to \infty$ limit, we proceed as before and replace each term with its appropriate limiting behaviour, and thus

$$
\begin{aligned}
\lim_{p \to \infty} \left| e_s^{\parallel}(p,t) \right| &\leq \varepsilon^2 \left(C_g C_s + \frac{C_s^2 C_f}{2} \right) e^{\lambda_s(t - T_s + p)} \int_{T_s}^t \frac{e^{\lambda_s(\tau - T_s + p)} + 2C_f \int_\tau^\infty e^{\lambda_s(\zeta - T_s + p)} e^{(\lambda_s + \lambda_u)(\tau - \zeta)} \mathrm{d}\zeta}{e^{2\lambda_s(\tau - T_s + p)}} \mathrm{d}\tau \\
&= \varepsilon^2 \left(C_g C_s + \frac{C_s^2 C_f}{2} \right) e^{\lambda_s(t - T_s + p)} \int_{T_s}^t \frac{e^{\lambda_s(\tau - T_s + p)} + 2C_f e^{\lambda_s(\tau - T_s + p)} e^{\lambda_u \tau} \int_\tau^\infty e^{-\lambda_u \zeta} \mathrm{d}\zeta}{e^{2\lambda_s(\tau - T_s + p)}} \mathrm{d}\tau \\
&= \varepsilon^2 \left(C_g C_s + \frac{C_s^2 C_f}{2} \right) e^{\lambda_s(t - T_s + p)} \int_{T_s}^t e^{-\lambda_s(\tau - T_s + p)} \left[1 + 2C_f e^{\lambda_u \tau} \int_\tau^\infty e^{-\lambda_u \zeta} \mathrm{d}\zeta \right] \mathrm{d}\tau \\
&= \varepsilon^2 \left(C_g C_s + \frac{C_s^2 C_f}{2} \right) e^{\lambda_s t} \int_{T_s}^t e^{-\lambda_s \tau} \left[1 + \frac{2C_f}{\lambda_u} \right] \mathrm{d}\tau \\
&= \varepsilon^2 \left(C_g C_s + \frac{C_s^2 C_f}{2} \right) \left(1 + \frac{2C_f}{\lambda_u} \right) e^{\lambda_s t} \frac{e^{-\lambda_s T_s} - e^{-\lambda_s t}}{\lambda_s} \\
&= \varepsilon^2 \frac{(2C_g C_s + C_s^2 C_f)(\lambda_u + 2C_f)}{2 |\lambda_s| \lambda_u} \left[1 - e^{\lambda_s(t - T_s)} \right]
\end{aligned}
$$

which is (27) as required. Thus, $e^{\parallel}$ remains $\mathcal{O}(\varepsilon^2)$ just as $e^{\perp}$ does, implying that $e_s(p,t)$ is $\mathcal{O}(\varepsilon^2)$ as desired.

5. Proof of Theorem 3.1

The proof is analogous to the stable manifold results, and requires the definitions

$$
\hat{M}_u^\varepsilon(p,t) := [Jf(x_u(t - T_u + p))]^T \frac{[x_u^\varepsilon(p,t) + e_u(p,t)] - x_u(t - T_u + p)}{\varepsilon} \quad (67)
$$

and

$$
\hat{B}_u^\varepsilon(p,t) := [f(x_u(t - T_u + p))]^T \frac{[x_u^\varepsilon(p,t) + e_u(p,t)] - x_u(t - T_u + p)}{\varepsilon}. \quad (68)
$$

The proof then proceeds exactly as in Theorem 2.1, with the only substantive changes being that the subscript s (for stable) needs to be replaced with the subscript u (for unstable), and that integration occurs from $-\infty$ to a general time as opposed to from a general time to $+\infty$ when working with the normal component of g. Details will not be provided.

6. Taylor-Green flow example

We present an example to demonstrate the efficacy of the theoretical method. Consider the Taylor-Green flow

$$
\left. \begin{aligned}
\dot{x} &= -\pi U \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi y}{L}\right) \\
\dot{y} &= \pi U \cos\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right)
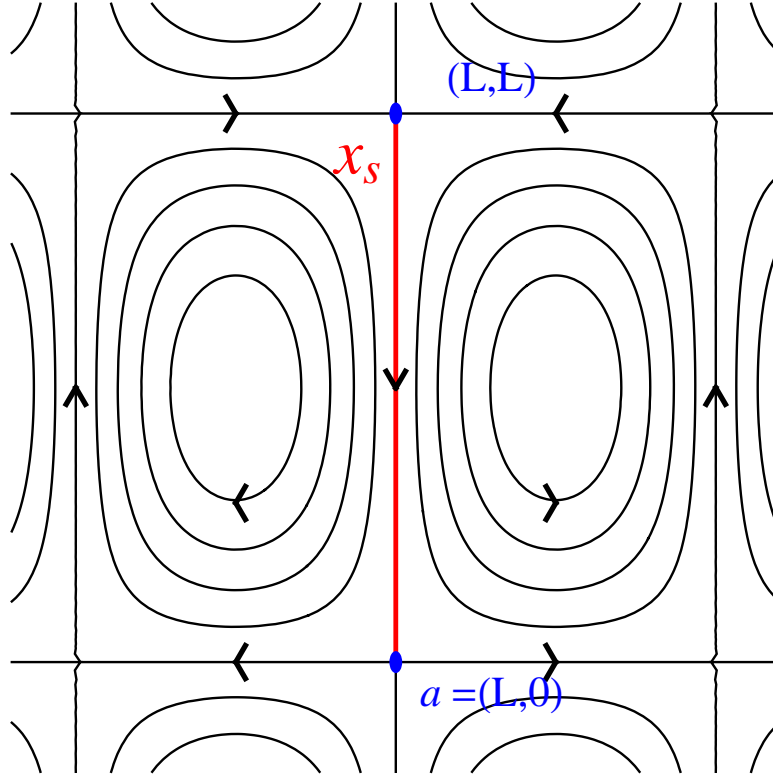\end{aligned} \right\}, \quad (69)
$$

Figure 7: The stable manifold branch of $(L, 0)$ in the Taylor-Green flow (69) which is to be controlled.

in which U and L are positive parameters. This flow is equivalent to the steady limit of the popular double-gyre model [7]. The autonomous system (69) possesses a heteroclinic trajectory from the fixed point (L, L) to that at $(L, 0)$, given by

$$x_{s,u}(t) = \begin{pmatrix} L \\ \frac{2L}{\pi} \tan^{-1} e^{-\pi^2 U t/L} \end{pmatrix},$$

which is shown in Fig. 7. Here, we will focus only on controlling the stable manifold x_s of the fixed point $a \equiv (L, 0)$. Since the manifold is downwards along the line $x = L$, the perpendicular and parallel components required in Def. 2.1 relate exactly to the x and $-y$ directions at every point on the heteroclinic. We shall try to move this stable manifold to the nonautonomous location

$$x_s^\varepsilon(p, t) = \begin{pmatrix} L \\ \frac{2L}{\pi} \tan^{-1} e^{-\pi^2 U(t - T_s + p)/L} \end{pmatrix} + \varepsilon L \begin{pmatrix} e^{-Up/L} \cos \frac{U(t-p)}{L} \\ 0 \end{pmatrix}, \quad (70)$$

23
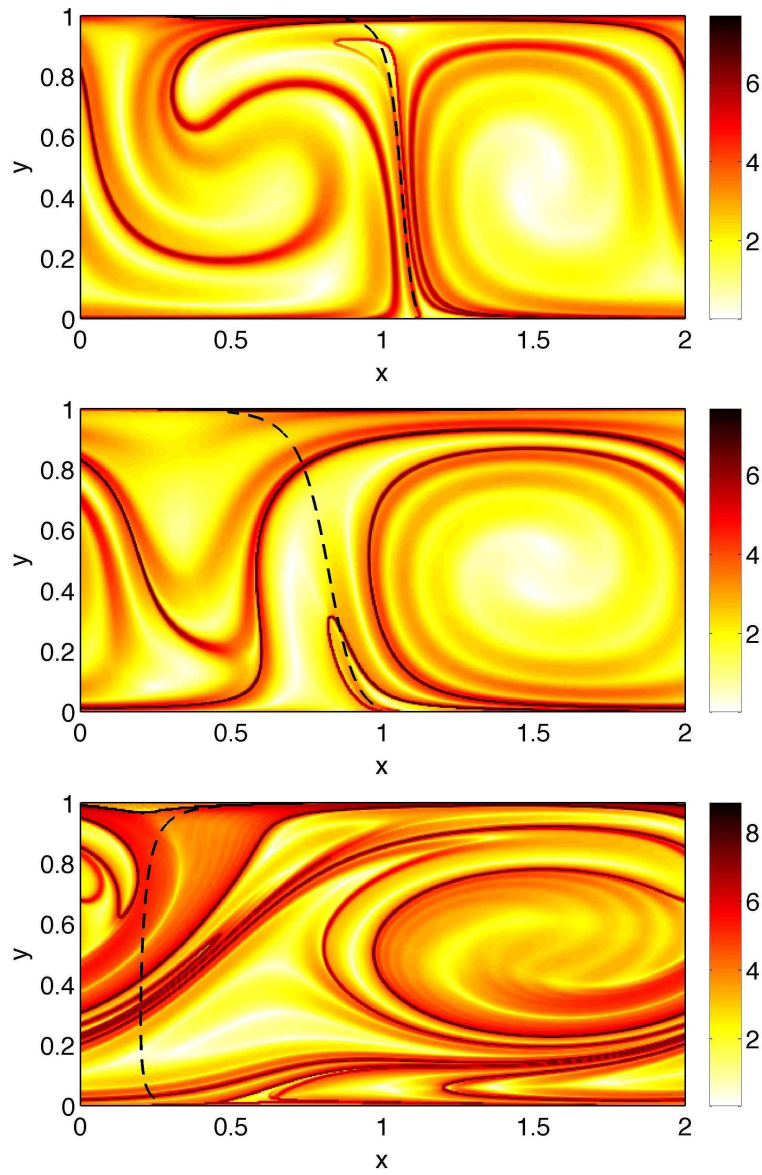
Figure 8: Finite-time Lyapunov exponent fields for (77) with $U = 1$, $L = 1$, $T_s = -1$ and $\varepsilon = 0.1$. The desired stable manifold (73) is shown by the black dashed curve, and the panels are respectively for the choices $t = 0.2$, 0.5, 1.5.
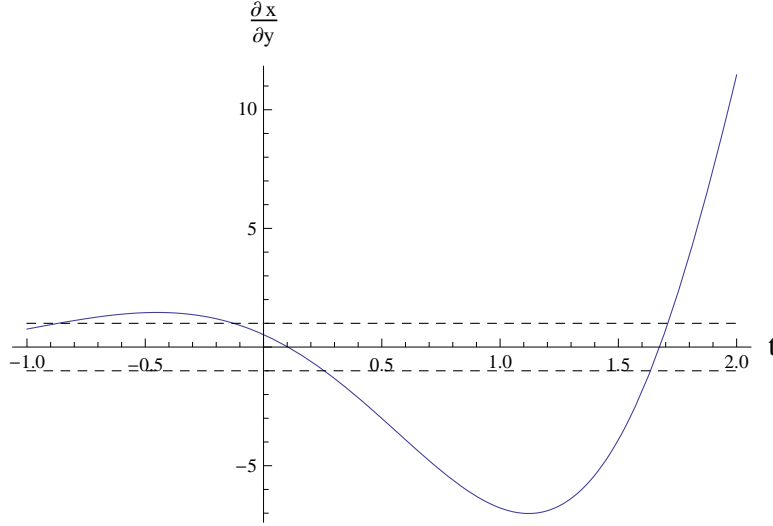
Figure 9: The slope $\partial x/\partial y$ of the manifold (73) in the limit $y \to 0$, for $U = 1$, $L = 1$, $T_s = -1$ and $\varepsilon = 0.1$.

by introducing a control velocity $g(x,y,t)$. The $\varepsilon = 0$ version of (70) is exactly $x_s(t - T_s + p)$. To determine the form of this curve in each time-slice t, we can think of (70) at each fixed t-value, subject to $t \geq T_s$. This would then be a parametric representation in terms of the parameter $p \geq S$; we can take $S_1(t) = S$ for all t and $S_2(t) = \infty$ for this chosen form. Thus, the theory will work on $\Xi_s = \{(p,t) \; : \; p \geq S \text{ and } t \geq T_s\}$. We can find the required stable manifold curve in each time-slice t by eliminating p from the parametric equation (70); since

$$y(p,t) = \frac{2L}{\pi} \tan^{-1} e^{-\pi^2 U(t - T_s + p)/L} \tag{71}$$

we have the relationship

$$p(y,t) = -\frac{L}{\pi^2 U} \ln\left(\tan \frac{\pi y}{2L}\right) + T_s - t\,, \tag{72}$$

and thus the stable manifold curve in each time-slice t in (x,y)-coordinates is

$$x = L\left\{1 + \varepsilon \exp\left[-\frac{Up(y,t)}{L}\right] \cos \frac{U\,(t - p(y,t))}{L}\right\}\,, \tag{73}$$

subject to the restrictions $t \geq T_s$ and $p \geq S$. The condition on p can be translated to

$$0 < y < y_m(t) := \frac{2L}{\pi} \tan^{-1} \exp\left[-\frac{\pi^2 U(t - T_s + S)}{L}\right] \tag{74}$$

where $y_m(t)$ is the maximum value of y attainable in the time-slice t. We observe that (70) also satisfies the congruence condition (13) since the $\mathcal{O}(\varepsilon)$ term in
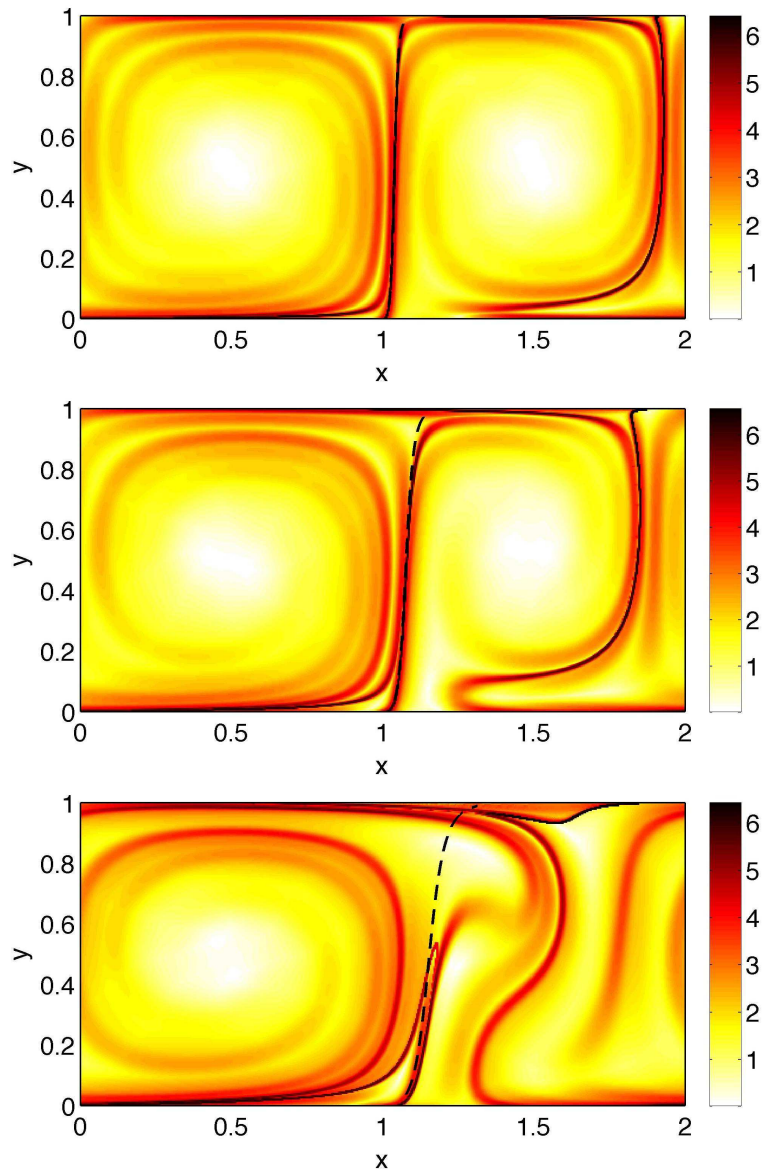
Figure 10: Finite-time Lyapunov exponent fields at $t = -0.9$ for (77) with $U = 1$, $L = 1$ and $T_s = -1$. The desired stable manifold (73) is shown by the black dashed curve, and the panels are respectively for the choices $\varepsilon = 0.05$, 0.1, 0.2.
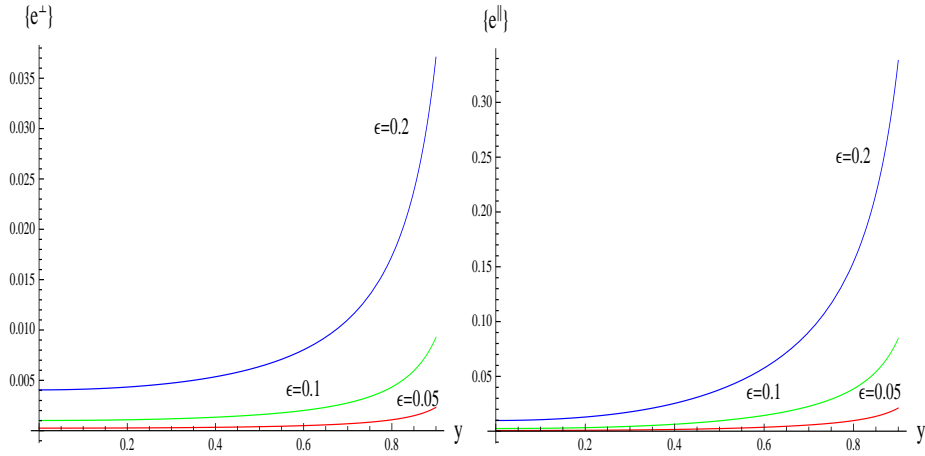
Figure 11: The y-variation of the squiggly bracketed terms in the perpendicular error (79) [left] and the parallel error (80) [right] bounds, at $t = -0.9$, $U = 1$, $L = 1$ and $T_s = -1$.

(70) is in the x-direction at all t, and is thus perpendicular to the unperturbed stable manifold. Now, in this case the components of the control $g(x, y, t)$ we need are $g^\perp$ (in the $+x$ direction) and $g^\parallel$ (in the $-y$ direction). By utilising the requirements in Def. 2.1 and doing the relevant algebra (not shown), we find that the control g needs to satisfy

$$g\left(L, y, t\right) = U \left(\begin{array}{l} -e^{-Up(y,t)/L} \left[\sin \frac{U(t-p(y,t))}{L} + \pi^2 \cos \frac{\pi y}{L} \cos \frac{U(t-p(y,t))}{L} \right] \\ \\ 0 \end{array} \right).$$

(75)

While any form for $g(x, y, t)$ consistent with (75) will result in our desired restricted stable manifold correct to $\mathcal{O}(\varepsilon)$, we choose

$$g\left(x, y, t\right) = U \left(\begin{array}{l} -e^{-Up(y,t)/L} \left[\sin \frac{U(t-p(y,t))}{L} + \pi^2 \cos \frac{\pi y}{L} \cos \frac{U(t-p(y,t))}{L} \right] \\ \\ \sin \frac{\pi x}{L} \sin \frac{Ut}{L} \end{array} \right),$$

(76)

which preserves incompressibility. Thus, the claim is that (73) is the restricted stable manifold of the system

$$\left. \begin{array}{l} \dot{x} = -\pi U \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi y}{L}\right) - \varepsilon U e^{-Up(y,t)/L} \left[\sin \frac{U(t-p(y,t))}{L} + \pi^2 \cos \frac{\pi y}{L} \cos \frac{U(t-p(y,t))}{L} \right] \\ \dot{y} = \pi U \cos\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) + \varepsilon U \sin \frac{\pi x}{L} \sin \frac{Ut}{L} \end{array} \right\},$$

(77)

in which $p(y, t)$ is given in (72).

In order to test the validity of the analytical results, we compare them with numerically approximated manifolds for the system (77). For this we approximate the respective finite-time Lyapunov (FTLE) fields, choosing an integration

interval $[t, t+1]$. Ridges in the FTLE field at t indicate—under certain additional assumptions [6]—the location of stable manifolds. We refer to [33] for a brief explanation of the computational scheme used in this paper. Recent work by Haller [6] sets the heuristical FTLE approach on a sound mathematical basis.

In Fig. 8, we examine the worsening of the control strategy with t, at a fixed ε. These and other experiments indicate that the control strategy works well in the range $T_s \leq t < T_s + 1.3$ for this example. Viewing the last panel ($t = 1.5$) in Fig. 8, we see that the mappability of the required stable manifold is being compromised near the hyperbolic point along the $y = 0$ line; the black dashed curve is becoming perpendicular to the unperturbed stable manifold $x = 1$. Mappability fails when absolute slope $|\partial x / \partial y|$ of the required manifold (73) goes to infinity in the required y domain. In the situation of (73) the absolute slope's maximum occurs in the limit $y \to 0$, and $|\partial x / \partial y| \to \infty$ as t gets larger. While $\lim_{y\to 0} \partial x / \partial y$ can be stated as an analytical expression in t, it is cumbersome, and hence we will show the slope variation graphically in Fig. 9 for the parameter values of Fig. 8. The dashed lines show when the slope is ± 1, which we use as a proxy for the slope getting too large, and find that this value is exceeded at $t \approx 0.3$, consistent with the performance observed in Fig. 8. This provides an *a priori* method for determining when mappability is close to failing.

In contrast, in Fig. 10 we show how the control method varies with ε at fixed $t = -0.9$, with $U = 1$, $L = 1$ and $T_s = -1$. This desired stable manifold matches up well with a ridge of the FTLE field when ε is sufficiently small, but worsens for larger y, reflecting the condition (74). Theorem 2.1 provides a method for determining the worsening with ε as seen in Fig. 10. We consider first $|e^\perp|$ as given in (24). Substituting the relevant f and x_s and simplifying leads to (calculations not shown)

$$\left| e^\perp(p,t) \right| \leq \varepsilon^2 \left[C_s C_g + \frac{C_s^2 C_f}{2} \right] \frac{2L}{\pi^2 U} \left\{ \frac{\frac{\pi}{4} - \tan^{-1}\left[\tanh \frac{\pi^2 U(t - T_s + p)}{2L} \right]}{\operatorname{sech} \frac{\pi^2 U(t - T_s + p)}{L}} \right\} . \quad (78)$$

In the above, the t and p occur not independently, but in the combination $t + p$; this is seen to always be the case so for (24) if $\operatorname{Tr} Df = 0$. Recasting in terms of the y-coordinate given in (71) leads to the expression

$$\left| e^\perp(y) \right| \leq \left[C_s C_g + \frac{C_s^2 C_f}{2} \right] \left\{ \varepsilon^2 \frac{2L}{\pi^2 U} \frac{\frac{\pi}{4} - \tan^{-1}\left[\tanh \frac{1}{2} \ln\left(\cot \frac{\pi y}{2L} \right) \right]}{\operatorname{sech}\left(\ln\left(\cot \frac{\pi y}{2L} \right) \right)} \right\} . \quad (79)$$

Similarly, we can use (26) to compute the bound for the parallel component of the error. This gives us

$$\left| e^\parallel(y,t) \right| \leq \left[C_s C_g + \frac{C_s^2 C_f}{2} \right] \left\{ \varepsilon^2 \operatorname{sech}\left[\ln\left(\cot \frac{\pi y}{2L} \right) \right] \int_{T_s}^{t} \left[\cosh\left[\ln\left(\cot \frac{\pi y}{2L} \right) + \frac{\pi^2 U(\tau - t)}{L} \right] \right. \right.$$

$$+ \frac{4 L C_f}{\pi^2 U} \cosh^2\left[\ln\left(\cot \frac{\pi y}{2L} \right) + \frac{\pi^2 U(\tau - t)}{L} \right] \left(\frac{\pi}{4} - \tan^{-1}\left(\tanh\left[\frac{1}{2} \ln\left(\cot \frac{\pi y}{2L} \right) + \frac{\pi^2 U(\tau - t)}{2L} \right] \right) \right) \right] d\tau \right\} (80)$$

in which the t-dependence persists. In this case from (3), we can set $C_f = \pi U \left(1 + \pi/L + (\pi/L)^2 \right)$. In Fig. 11, we show the two terms in squiggly brackets in (79) and (80), at the parameter values of Fig. 10. While acknowledging that these terms are associated with *bounds* for the error rather than the error itself, we see from Fig. 11 exactly the behaviour we anticipate: rapid error growth with y at each ε, and amplification of error with ε at each y-value. In the main, however, we note from this example that we are able to verify that the control method works very well for sufficiently small ε.

7. Concluding remarks

We have in this article developed a theoretical framework based on which it is possible to move a stable/unstable manifold in a two-dimensional autonomous system, to a *desired* nonautonomous location which is subjected to certain mappability conditions. A rigorous error estimate for the procedure was developed. A numerical example was used to demonstrate the efficacy of the manifold control method. To our knowledge, this is the first study which furnishes a method for controlling stable and unstable manifolds nonautonomously in the sense of making them follow a user-specified time-variation.

In a forthcoming article, we will develop methods for simplifying the hypotheses required for the restricted stable and unstable manifolds, in order to address the computationally natural situation of attempting to achieve a desired stable/unstable manifold which is given in the form $f(x, y, t) = 0$, as opposed to having to work through the parameter p. Preliminary results indicate that the control strategy can be implemented, for example, to achieve highly wiggly user-specified nonautonomous invariant manifolds. We expect to obtain insights into a more natural implementation of the mappability condition, so that unreasonable expectations from our control strategy (such as the dashed curve we tried to require in the final panel in Fig. 8) are avoided. Extensive numerical analyses will be performed in all these situations.

Given the considerable interest in the literature in quantifying transport due to the breaking of homo/heteroclinic separatrices [22, 7, 24, 25, 28, 29, 12, 13, 34, 32], an interesting question would be whether it would be possible to *independently* move previously coexisting stable and unstable manifolds to separate user-defined time-varying locations using the methods developed in this article. In this situation $x_s = x_u$, which we shall denote by x_σ. Now examining the definitions for the leading-order control velocities, we see that this would potentially give conflicting instructions on what $g\left(x_\sigma(t - T_{s,u} + p), t\right)$ needs to be. Apparently, achieving such "double" control requires higher-order information—or perhaps an alternative viewpoint—which will be pursued in the future.

This article complements the authors' work on controlling hyperbolic trajectories (that is, the "beginning of stable/unstable manifolds"). In ongoing research, recent two-dimensional control strategies [35] are being extended to arbitrary dimensions, and to arbitrarily high-order accuracy, for such hyperbolic

trajectories. Building on the present article, we plan to similarly extend control strategies to stable/unstable manifolds in high dimensions, and also pursue accuracy to higher-order in ε.

[1] G. Haller, F. Beron-Vera, Geodesic theory of transport barriers in two-dimensional flows, Phys. D 241 (2012) 1680–1702.

[2] D. Blazevski, G. Haller, Hyperbolic and elliptic transport barriers in three-dimensional unsteady flows (2013). arXiv: submit/0752347 (submitted).

[3] M. Allshouse, J.-L. Thiffeault, Detecting coherent structures using braids, Phys. D 241 (2012) 95–105.

[4] G. Froyland, K. Padberg, Finite-time entropy: a probabilistic method for measuring nonlinear stretching, Phys. D 241 (2012) 1612–1628.

[5] M. Budišić, I. Mezić, Geometry of ergodic quotient reveals coherent structures in flows, Phys. D 241 (2012) 1255–1269.

[6] G. Haller, A variational theory for Lagrangian Coherent Structures, Phys. D 240 (2011) 574–598.

[7] S. Shadden, F. Lekien, J. Marsden, Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows, Phys. D 212 (2005) 271–304.

[8] M. Dellnitz, O. Junge, Almost invariant sets in Chua's circuit, Int. J. Bif. Chaos 7 (1997) 2475–2485.

[9] G. Froyland, S. Lloyd, N. Santitissadeekorn, Coherent sets for nonautonomous dynamical systems, Phys. D 239 (2010) 1527–1541.

[10] G. Froyland, N. Santitissadeekorn, A. Monahan, Transport in time-dependent dynamical systems: finite-time coherent sets, Chaos 20 (2010) 043116.

[11] I. Mezić, S. Loire, V. Fonoberov, P. Hogan, A new mixing diagnostic and Gulf oil spill movement, Science 330 (2010) 486–489.

[12] S. Balasuriya, A tangential displacement theory for locating perturbed saddles and their manifolds, SIAM J. Appl. Dyn. Sys. 10 (2011) 1100–1126.

[13] S. Balasuriya, Nonautonomous flows as open dynamical sytems: characterising escape rates and time-varying boundaries, in: Ergodic Theory, Open Dynamics and Structures, Springer, 2014. In press.

[14] S. Balasuriya, Explicit invariant manifolds and specialised trajectories in a class of unsteady flows, Phys. Fluids 24 (2012) 127101.

[15] M. Farazmand, G. Haller, Computing Lagrangian Coherent Structures from variational LCS theory, Chaos 22 (2012) 013128.

[16] T. Peacock, J. Dabiri, Introduction to focus issue: Lagrangian Coherent Structures, Chaos 20 (2010) 017501.

[17] G. Boffetta, G. Lacorata, G. Radaelli, A. Vulpiani, Detecting barriers to transport: a review of different techniques, Phys. D 159 (2001) 58–70.

[18] B. Krauskopf, H. Osinga, E. Doedel, M. Henderson, J. Guckenheimer, A. Vladimirsky, M. Dellnitz, O. Junge, A survey of method's for computing (un)stable manifold of vector fields, Int. J. Bif. Chaos 15 (2005) 763–791.

[19] G. Froyland, An analytic framework for identifying finite-time coherent sets in time-dependent dynamical systems, Phys. D 250 (2013) 1–19.

[20] V. K. Melnikov, On the stability of the centre for time-periodic perturbations, Trans. Moscow Math. Soc. 12 (1963) 1–56.

[21] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Springer, New York, 1983.

[22] V. Rom-Kedar, A. Leonard, S. Wiggins, An analytical study of transport, mixing and chaos in an unsteady vortical flow, J. Fluid Mech. 214 (1990) 347–394.

[23] S. Wiggins, Chaotic Transport in Dynamical Systems, Springer-Verlag, New York, 1992.

[24] S. Balasuriya, Direct chaotic flux quantification in perturbed planar flows: general time-periodicity, SIAM J. Appl. Dyn. Sys. 4 (2005) 282–311.

[25] S. Balasuriya, Cross-separatrix flux in time-aperiodic and time-impulsive flows, Nonlinearity 19 (2006) 282–311.

[26] T. Radko, On the generation of large-scale structures in a homogeneous eddy field, J. Fluid Mech. 668 (2011) 76–99.

[27] S. Chandrasesekhar, Hydrodynamics and hydrodynamic stability, Dover, New York, 1961.

[28] S. Balasuriya, Approach for maximizing chaotic mixing in microfluidic devices, Phys. Fluids 17 (2005) 118103.

[29] S. Balasuriya, M. Finn, Energy constrained transport maximization across a fluid interface, Phys. Rev. Lett. 108 (2012) 244503.

[30] W. A. Coppel, Dichotomies in Stability Theory, number 629 in Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1978.

[31] Y. Yi, A generalized integral manifold theorem, J. Differential Equations 102 (1993) 153–187.

[32] K. Yagasaki, Invariant manifolds and control of hyperbolic trajectories on infinite- or finite-time intervals, Dyn. Sys. 23 (2008) 309–331.

[33] K. Padberg, T. Hauff, F. Jenko, O. Junge, Lagrangian structures and transport in turbulent magnetized plasmas, New J. Phys. 9 (2007) 400.

[34] S. Balasuriya, Optimal frequency for microfluidic mixing across a fluid interface, Phys. Rev. Lett. 105 (2010) 064501.

[35] S. Balasuriya, K. Padberg-Gehle, Controlling the unsteady analogue of saddle stagnation points, SIAM J. Appl. Math. 73 (2013) 1038–1057.