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Vanishing Viscosity in the Barotropic β-Plane

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The initial boundary value problems associated with the inviscid barotropic potential vorticity equation in the β-plane and its viscous analogue are considered. It is shown that the solution velocity to the viscous equation converges to the inviscid solution in a C^1 sense for finite times and that, under additional smoothness assumptions on the inviscid flow, this convergence can be extended to C^3. Moreover, this convergence occurs as $O(\varepsilon)$, where ε is the viscous parameter. This particular form of vanishing viscosity is of relevance in analysing viscosity induced advection for barotropic models.

1. THE INITIAL BOUNDARY VALUE PROBLEMS

This paper establishes a vanishing viscosity result for geophysical flows which satisfy the barotropic β-plane potential vorticity equation [1, 2]. This will be investigated for several norms of interest, as will be the rate of such convergence with respect to the viscosity. Our motivation for deriving vanishing viscosity estimates in this precise form is to explain viscosity induced advection in barotropic models for the Gulf Stream [3, 4]. For a more detailed exposition of these results, the reader is referred to [3, Chap. 4]. The barotropic assumption removes the vertical dimension from the geophysical equations [1, 2], resulting in a flow which lies on a two-dimensional surface which we label Ω. In the context of the physical problem Ω is described by the variables (x, y), corresponding to the local eastward and northward coordinates, respectively [1, 2]. Under certain assumptions, the inviscid flow on Ω can be represented by the conservation of barotropic (β-plane) potential vorticity

$$\frac{Dq}{Dt} = 0,$$

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where the material derivative $D/Dt = \partial/\partial t - (\partial\psi/\partial y)(\partial/\partial x) + (\partial\psi/\partial x)(\partial/\partial y)$, and the potential vorticity $q(x, y, t)$ is given by $q(x, y, t) = \Delta\psi + \beta y$ [1, 2]. Here, $\psi(x, y, t)$ is the streamfunction associated with the flow, and Δ is the Laplacian in the (x, y) variables. The Coriolis parameter β is a positive constant. We may write the conservation equation entirely in terms of the streamfunction as $\Delta\psi_t + \{\psi, \Delta\psi + \beta y\} = 0$, by using the Poisson bracket notation $\{f, g\} = (\partial f/\partial x)(\partial g/\partial y) - (\partial f/\partial y)(\partial g/\partial x)$. If, on the other hand, we include the effect of viscosity in the potential vorticity conservation equation, we obtain a dissipative term $\varepsilon\Delta q$ on the right hand side, where ε is a positive parameter representing the viscosity, or, alternatively, is the reciprocal of the Reynolds number. We now pose initial boundary value problems associated with the inviscid and viscous barotropic potential vorticity equations. Let $\psi^0(x, y, t)$ be the solution to the initial boundary value problem (IBVP) corresponding to the inviscid potential vorticity equation. Hence, ψ^0 satisfies

$$\left.\begin{aligned} &\Delta\psi_t^0 + \{\psi^0, \Delta\psi^0 + \beta y\} = 0 \quad \text{in } \Omega \\ &\psi^0|_{\partial\Omega} = 0 \\ &\psi^0|_{t=0} = \phi(x, y). \end{aligned}\right\} \tag{1}$$

Similarly, let $\psi(x, y, t; \varepsilon)$ be the solution to the initial boundary value problem of the viscous potential vorticity equation, which can be posed as

$$\left.\begin{aligned} &\Delta\psi_t + \{\psi, \Delta\psi + \beta y\} = \varepsilon\Delta^2\psi \quad \text{in } \Omega \\ &\psi|_{\partial\Omega} = \Delta\psi|_{\partial\Omega} = 0 \\ &\psi|_{t=0} = \phi(x, y). \end{aligned}\right\} \tag{2}$$

The viscous parameter ε is assumed to be in $I = (0, \varepsilon_0]$, where ε_0 is sufficiently small, and ϕ is a suitably smooth initial condition. Our objective is to show that the solution of (2) converges in some sense to the solution of (1) as ε goes to zero. In particular, we will be interested in the particular speed of such convergence, and will show that, for certain norms of interest, the convergence to zero occurs as fast as ε. Our primary goal will be showing that for any $T > 0$, there exists a constant $C(T)$ such that

$$\sup_{t\in[0,T]} \left\| \nabla\big(\psi(x, y, t; \varepsilon) - \psi^0(x, y, t)\big)\right\|_{C^3(\Omega)} \leq C(T)\varepsilon, \tag{3}$$

where ∇ is the gradient in the (x, y) variables.

The result above is not an end in itself; it is an important intermediate step in addressing viscous transport in barotropic β-plane models for oceanic jets such as the Gulf Stream [3, 4]. It is necessary in these works to use the fact that the singular perturbation provided by the viscosity in (2) provides only a regular perturbation in the Eulerian velocity field for finite times. Moreover, the particular form (3) is required for the use of a Melnikov approach in computing distances between perturbed manifolds and predicting induced advection [3, 4].

For Eqs. (1) and (2), existence and smoothness results are well known for the case where $\beta = 0$, i.e., when geophysical effects are ignored. The interested reader is referred to the original papers [5–13, 16] or to summaries in [14, 15]. For two-dimensional unsteady flow with no boundary and sufficiently smooth initial condition, solutions to the IBVPs (1) and (2) exist for all $t > 0$, and are unique and smooth. For example, Ebin and Marsden show that for each finite t, the solution is in $C^2(\Omega) \cap H^m(\Omega)$. For three-dimensional flows, however, such results do not, in general, exist.

The barotropic equations (1) and (2) differ from the non-geophysical vorticity equations only in the presence of the additional term $\beta\psi_x$. This is linear with a constant coefficient, and of a lower order than the highest derivatives. Hence, the theory is expected to extend trivially to the equations on the β-plane as given in (1) and (2). In fact, Bourgeois and Beale have shown the existence of solutions when $\beta \neq 0$ in three dimensions with periodic horizontal boundary conditions and rigid lid and bottom [17].

In the case where $\beta = 0$, in addition to existence and smoothness results, some vanishing viscosity limits have also been established for a certain class of Ω. Ladyzhenskaya shows that convergence occurs in the L^2 norm [8]. We will closely follow her arguments in establishing an L^2 convergence for the nonzero β case. Ebin and Marsden use techniques from differential geometry to show that the solutions converge in the H^m norm for $\beta = 0$ [11]. A fixed point argument is employed by McGrath to show L^1 convergence [12], while embedding theorems constitute the main arguments in Golovkin's proof of convergence of the velocity fields in the C^1 norm [13]. Kato and Ponce prove convergence in the Lebesgue spaces L_s^p using more direct estimates [16]. It seems reasonable to expect quick extensions of these results to the $\beta \neq 0$ case; however, many of the proof strategies cannot be modified easily. Moreover, it is not obvious from these methods that an estimate in the form (3) can be derived, as is specifically required for predicting transport in barotropic jets [3, 4].

We are particularly interested in proving $O(\varepsilon)$ convergence of the difference in the Eulerian velocity fields. Once again, for $\beta = 0$, certain convergence rates are well known, but not necessarily in the precise norms which we require. If the initial condition is smooth enough, an $O(\varepsilon)$ convergence occurs in the L^2 norm [8]. If, however, the initial condition is not smooth, this convergence may be as slow as $\sqrt{\varepsilon}$ [18, 19]. For smooth initial conditions, we are able to establish $O(\varepsilon)$ convergence for the C^0 norm when $\beta \neq 0$, using direct estimates and the Sobolev Embedding Theorem. Under additional assumptions on the smoothness of ψ^0, we establish the same result in $C^3(\Omega)$. This final result is precisely that required to validate the Melnikov calculations in [3, 4], in which viscous transport in barotropic jets is examined.

2. NOTATION, HYPOTHESES, AND RELEVANT RESULTS

Here, we summarise our notation, hypotheses, and some existing results which will be of use in subsequent sections.

DEFINITION 1. The $L^2(\Omega)$ norm of a measurable function f: $\Omega \to \mathbb{R}^2$ is defined by

$$\|f\| \equiv \left[\int_\Omega f(x, y) \cdot f(x, y)\, dx\, dy\right]^{1/2}.$$

DEFINITION 2. Let v: $\Omega \to \mathbb{R}^2$, be given in component form as $v = (v_1, v_2)$. The $L^\infty(\Omega)$ norm of v is defined by $\|v\|_\infty \equiv \|v_1\|_\infty + \|v_2\|_\infty$.

DEFINITION 3. Let v: $\Omega \to \mathbb{R}$, f: $\mathbb{R} \to \mathbb{R}$, and $m \in \mathbb{Z}^+$, where Ω is two-dimensional, and described by the variables (x, y). Then, the multi-index notation $\sum_{|k|=m} f(D^k v)$ represents the sum of f over the $(m+1)$ derivatives of ψ of order m in x and y.

When the norm symbols $\|\cdot\|$ are used with no subscript, the $L^2(\Omega)$ norm is understood. Whenever generalised derivative symbols ∇, Δ, and D^k are used, these pertain only to the spatial variables x and y. All subsequent derivations in this paper will implicitly assume that the following hypotheses concerning the domain Ω and the initial condition $\phi(x, y)$ are satisfied.

HYPOTHESIS 1. *The domain Ω is smooth, connected, two-dimensional, and has no boundary, i.e., $\partial\Omega = \emptyset$.*

HYPOTHESIS 2. *The initial condition $\phi(x, y)$ is assumed to be smooth enough so that*

$$\|\nabla\phi\|,\ \|\Delta\phi\|,\ \|\nabla(\Delta\phi)\|,\ \|\Delta^2\phi\|,\ \sup_\Omega\left\{|\phi_x| + |\phi_y|\right\},\ \sup_\Omega|\Delta\phi|,\ \textit{and}\ \|\Delta^2\phi\|_4$$

are all finite.

Examples of sets with no boundary are a doubly periodic domain (a torus), a singly periodic domain which is unbounded in the other direction (an infinite cylinder), and $\mathbb{R}^2$ itself. The unbounded β-plane is ostensibly $\mathbb{R}^2$, but in the frequently considered case of periodic boundary conditions in the x direction, may be taken to be the cylinder infinite in the y direction. By the boundary condition $\psi|_{\partial\Omega} = 0$ for an unbounded domain Ω we mean that the function ψ must decay to zero in the unbounded directions. For a bounded domain the boundary conditions in Eqs. (1) and (2) are vacuous.

We are interested in estimating quantities uniformly for $t \in [0, T]$, independently of $\varepsilon \in I$, and thus we use the notation C_i, $i \in \mathbb{Z}^+$, to denote generic positive constants depending only upon β, T and the finite norms of the initial condition ϕ stated in Hypothesis 2. In particular, the C_i's will be independent of ε. The values of these C_i's will only be consistent within each lemma. We now briefly present a few results, due to Golovkin [13], which will be of use in our subsequent derivations.

PROPOSITION 1. *Suppose Ω is two-dimensional and smooth with no boundary, and that $v: \Omega \to \mathbb{R}$. If $1 \le p < \infty$, there exists a positive constant $C_1(p)$ independent of v such that*

$$\sum_{|k|=2} \|D^k v\|_p \le C_1(p)\|\Delta v\|_p \tag{4}$$

whenever the right hand side is bounded. Similarly, there exist positive constants C_2 and C_3 independent of v such that

$$\sum_{|k|=2} \|D^k v\|_\infty \le C_2\|\Delta v\|_\infty\{1 + |\log\|\Delta^2 v\|\,| + |\log\|\Delta v\|_\infty|\} + C_3 \sum_{|k|=1} \|D^k v\| \tag{5}$$

whenever the norms on the right hand side make sense.

Proof. The reader is to referred to the paper by Golovkin for these proofs [13]. Inequality (4) is derived in his Lemma 2, and is an improvement on the standard regularity of elliptic operators results, while (5) is a special case of his Lemmas 5 and 6 (please note the typographic error in the final norm of his Lemma 6). ∎

PROPOSITION 2. *Let Ω be two-dimensional and smooth with no boundary, and let $v: \Omega \to \mathbb{R}$. Then, for any $\delta > 0$, there exists a positive constant C_δ such that*

$$\sum_{1 \le |k| \le 2} \|D^k v\|_\infty \le \delta\left(\sum_{|k|=1} \|D^k v\| + \|\Delta^2 v\|\right) + C_\delta \sum_{|k|=1} \|D^k v\| \tag{6}$$

whenever the norms on the right hand side are bounded.

Proof. This is a special case of Lemma 9 in [13]. ▮

3. A PRIORI ESTIMATES

We now assume that smooth enough solutions to (1) and (2) exist in $Q_T = \Omega \times [0, T]$ where $T \in (0, \infty)$, and derive bounds on derivatives of the solutions in relevant norms. The proofs in this section are straightforward but lengthy. In the interest of brevity, we may suppress the t dependence of functions, and also neglect to specify the differential elements in integrals; the relevant domains of integration will be obvious from the context.

LEMMA 1 (Energy Equality). *Let $\psi(x, y, t; \varepsilon)$ satisfy (2) in Q_T. Then, for any $\tau \in [0, T]$,*

$$\|\nabla\psi(\tau)\|^2 + 2\varepsilon\int_0^\tau \|\Delta\psi(t)\|^2\,dt = \|\nabla\phi\|^2.$$

Proof. This proof is a modified version of that by Ladyzhenskaya [8]. Multiply (2) by ψ to get

$$\psi\Delta\psi_t + \psi\psi_x(\Delta\psi_y + \beta) - \psi\psi_y\Delta\psi_x - \varepsilon\psi\Delta^2\psi = 0, \tag{7}$$

which is assumed valid for any $t \in [0, T]$. We now apply the operator $\int_\Omega dx\,dy$ to the above. We handle the terms individually. By integrating by parts and noting that the boundary terms disappear since $\psi|_{\partial\Omega} = 0$, we obtain $\int_\Omega \psi\Delta\psi_t = \int_\Omega \psi(\psi_{xxt} + \psi_{yyt}) = -\int_\Omega \psi_x\psi_{xt} - \int_\Omega \psi_y\psi_{yt}, = -(1/2)(d/dt)\|\nabla\psi\|^2$. Also, $\int_\Omega \beta\psi\psi_x = \int_\Omega(\beta/2)(\psi^2)_x = 0$ by the fact that ψ disappears on the boundary of Ω. The next term of (7) is simplified by integrating by parts as

$$\int_\Omega (\psi\psi_x\Delta\psi_y - \psi\psi_y\Delta\psi_x) = \int_\Omega \left[\left(\frac{\psi^2}{2}\right)_x \Delta\psi_y - \left(\frac{\psi^2}{2}\right)_y \Delta\psi_x\right] = 0$$

since the boundary terms are zero, and the other two terms cancel. Integrating by parts twice, and noting that the boundary terms involve $\Delta\psi$ and ψ and thus disappear, leads to $-\varepsilon\int_\Omega \psi\Delta^2\psi = -\varepsilon\|\Delta\psi\|^2$. Substituting all these simplifications in (7), we obtain $(d/dt)\|\nabla\psi\|^2 + 2\varepsilon\|\Delta\psi\|^2 = 0$. The result follows by integrating this expression over time from 0 to τ. ▮

LEMMA 2. *If $\psi(x, y, t; \varepsilon)$ satisfies (2) in Q_T, then for any $t \in [0, T]$, $\sum_{|k|=2}\|D^k\psi(t)\| \le C_1$.*

Proof. Again, we closely follow Ladyzhenskaya [8]. Multiply (2) by $\Delta\psi$ to get

$$(\Delta\psi)(\Delta\psi_t) + (\Delta\psi)\psi_x(\Delta\psi_y) - (\Delta\psi)\psi_y(\Delta\psi_x) + \beta(\Delta\psi)\psi_x - \varepsilon(\Delta\psi)(\Delta^2\psi) = 0 \tag{8}$$

for each $t \in [0,T]$. As in the proof of Lemma 1, we apply the operator $\int_\Omega dx\,dy$ to the above, and keep track of its effects on each term separately. Firstly, we have $\int_\Omega(\Delta\psi)(\Delta\psi_t) = (1/2)(d/dt)\|\Delta\psi\|^2$. Note from Lemma 1 that $\nabla\psi$ is square integrable over Ω. If Ω is unbounded, this means that $\nabla\psi$ must go to zero on $\partial\Omega$. Thus, it is apparent that the next two terms yield zero by integrating by parts. The β term also integrates to zero, since $\beta\int_\Omega \psi_x\psi_{xx} = -\beta\int_\Omega \psi_{xx}\psi_x$ by integrating by parts, and thus must equal zero. A similar argument holds for the term $\beta\psi_x\psi_{yy}$. The term involving ε also admits to integration by parts, to yield $-\varepsilon\int_\Omega \Delta\psi\,\Delta^2\psi = \varepsilon\|\nabla(\Delta\psi)\|^2$. Substituting these integrals into the integrated version of (8), we obtain $(1/2)(d/dt)\|\Delta\psi(t)\|^2 + \varepsilon\|\nabla(\Delta\psi)\|^2 = 0$, which when integrated yields the fact that the Laplacian of ψ satisfies $\|\Delta\psi(\tau)\|^2 + 2\varepsilon\int_0^\tau\|\nabla(\Delta\psi)\|^2\,dt = \|\Delta\phi\|^2$. Using (4), this implies that all second spatial derivatives of ψ are bounded for finite time. ∎

LEMMA 3. *Let $\psi(x,y,t;\varepsilon)$ satisfy* (2) *in Q_T. Then, for each $p \geq 2$, there exists a positive constant $C_1(p)$ such that for each $t \in [0,T]$, $\sum_{|k|=1}\|D^k\psi(t)\|_p \leq C_1(p)$.*

Proof. Fix $t \in [0,T]$, and consider the function $\sum_{|k|=1}\psi(x,y,t;\varepsilon)$, which is in the Sobolev space $W^{1,2}(\Omega)$ by Lemmas 1 and 2. The limiting $(n - mp = 0)$ version of the Sobolev Embedding Theorem implies the existence of a positive constant $C_2(p)$ for each $p \geq 2$ such that $\|\psi_x(t)\|_p \leq C_2(p)$. ∎

LEMMA 4. *Let $\psi(x,y,t;\varepsilon)$ satisfy* (2) *in Q_T. Then, there exists a positive constant C_3 such that $\sum_{|k|=1}\|D^k\psi(t)\|_{C^0} \leq C_3$ for each $t \in [0,T]$.*

Proof. The result of Lemma 3 indicates that $\nabla\psi$ is bounded in $[0,T]$ in the $L^4(\Omega)$ norm in the sense that there exists a constant C_4 such that

$$\|\psi_x\|_4 + \|\psi_y\|_4 \leq C_4. \tag{9}$$

We now show that $\Delta\psi$ is also bounded in L^4, using the same technique as in the proof of Lemma 2. Multiply (2) by $(\Delta\psi)^3$ to get for each t,

$$(\Delta\psi)^3(\Delta\psi_t) + (\Delta\psi)^3\psi_x(\Delta\psi_y) - (\Delta\psi)^3\psi_y(\Delta\psi_x) + \beta(\Delta\psi)^3\psi_x - \varepsilon(\Delta\psi)^3(\Delta^2\psi) = 0. \tag{10}$$

We integrate (10) over Ω as before, and handle the terms individually. The combination of the second and third term integrates to zero by parts, and we obtain

$$\frac{1}{4}\frac{d}{dt}\|\Delta\psi\|_4^4 + 3\varepsilon\int_\Omega (\Delta\psi)^2\left[(\Delta\psi_x)^2 + (\Delta\psi_y)^2\right] = -\beta\int_\Omega \psi_x(\Delta\psi)^3.$$

Since the term involving ε is nonnegative, $(1/4)(d/dt)\|\Delta\psi\|_4^4 \leq \beta\int_\Omega|\psi_x(\Delta\psi)^3| \leq \beta\|\psi_x\|_4\|\Delta\psi\|_4^3$ by using Hölder's inequality. Therefore, $(d/dt)\|\Delta\psi\|_4 \leq \beta\|\psi_x\|_4$, and since, by (9), $\|\psi_x\|_4$ is bounded, there exists C_2 such that $\|\Delta\psi\|_4 \leq C_2$. However, by (4) this implies that all second spatial derivatives of ψ are bounded in the L^4 norm. This, together with (9), permits the use of the Sobolev Embedding Theorem [20] once again on the function ψ_x (resp. ψ_y), which is in $W^{1,4}(\Omega)$, to obtain the result in $C^0(\Omega)$. ∎

LEMMA 5. *Let $\psi(x, y, t; \varepsilon)$ satisfy (2). Then there exists a positive constant C_1 such that*

$$\sup_{Q_T}|\Delta\psi(x, y, t)| \leq C_1.$$

Proof. The proof uses elements from the maximum principle argument of Ladyzhenskaya [8]. Let $\hat{w} = \Delta\psi e^{-t}$. Then, from (2),

$$\left.\begin{aligned} \hat{w}_t - \varepsilon\Delta\hat{w} + \{\psi, \hat{w}\} + \hat{w} &= -\beta\psi_x e^{-t}, \\ \hat{w}|_{\partial\Omega} &= 0, \\ \hat{w}(x, y, 0) &= \Delta\phi(x, y). \end{aligned}\right\}$$

Suppose the positive maximum of $\hat{w}$ in Q_T does not lie in $\{t = 0\}$. It cannot lie on $\partial\Omega$, because of the boundary condition, and thus must be in $\Omega \times (0, T]$. At such a maximum, $\hat{w}_t \geq 0$, $-\varepsilon\Delta\hat{w} \geq 0$, and $\hat{w}_x = \hat{w}_y = 0$. Thus $\hat{w} \leq -\beta\psi_x e^{-t}$ within $\Omega \times (0, T]$. If the negative minimum of $\hat{w}$ also lies in the same set, by the same argument, $\hat{w}_t \leq 0$, $-\varepsilon\Delta\hat{w} \leq 0$, $\hat{w}_x = \hat{w}_y = 0$, and therefore $\hat{w} \geq -\beta\psi_x e^{-t}$. Thus, at all points in Q_T, $|\hat{w}| \leq \max\{\sup_\Omega|\Delta\phi|, \sup_{Q_T}|\beta\psi_x e^{-t}|\}$, and $|\Delta\psi| \leq \max\{\sup_\Omega|\Delta\phi e^T|, \sup_{Q_T}|\beta\psi_x|\}$. However, $\sup_\Omega|\Delta\phi|$ is bounded by Hypothesis 2, and $\beta\sup_{Q_T}|\psi_x|$ is bounded independent of ε by Lemma 4. Thus, $|\Delta\psi|$ is itself bounded in Q_T. ∎

LEMMA 6. *Let ψ satisfy (2) in Q_T. Then, there exist positive constants C_4, C_5, and C_6 such that for any $t \in [0, T]$,*

$$\frac{1}{2}\frac{d}{dt}\|\nabla(\Delta\psi)\|^2 + \varepsilon\|\Delta^2\psi\|^2$$

$$\leq \beta C_4\|\nabla(\Delta\psi)\| + \left(C_5 + C_6|\log\|\Delta^2\psi\||\right)\|\nabla(\Delta\psi)\|^2. \qquad (11)$$

Proof. Apply the gradient operator ∇ to (2) and then take the dot product with $\nabla(\Delta\psi)$. This results in

$$\nabla(\Delta\psi)\cdot\nabla(\Delta\psi)_t + \left(\Delta\psi_y\,\nabla\psi_x - \Delta\psi_x\,\nabla\psi_y\right)\cdot\nabla(\Delta\psi) + \beta\nabla(\psi_x)\cdot\nabla(\Delta\psi)$$
$$+\left(\psi_x\nabla(\Delta\psi_y) - \psi_y\nabla(\Delta\psi_x)\right)\cdot\nabla(\Delta\psi) = \varepsilon\nabla(\Delta^2\psi)\cdot\nabla(\Delta\psi).$$

As before, we apply the operator $\int_\Omega$ to the above, and tackle the terms individually. The first term is easily seen to yield the first term of (11), while the next term is bounded by

$$\left|\int_\Omega\left(\Delta\psi_y\,\nabla\psi_x - \Delta\psi_x\,\nabla\psi_y\right)\cdot\nabla(\Delta\psi)\right|$$
$$=\left|\int_\Omega\left[\Delta\psi_x\,\Delta\psi_y(\psi_{xx}-\psi_{yy}) + \psi_{xy}\left((\Delta\psi_y)^2-(\Delta\psi_x)^2\right)\right]\right|$$
$$\le C_7\sup_{Q_T}\left\{|\psi_{xx}|+|\psi_{xy}|+|\psi_{yy}|\right\}\|\nabla(\Delta\psi)\|^2$$
$$\le\left(C_8 + C_9|\log\|\Delta^2\psi\||\right)\|\nabla(\Delta\psi)\|^2,$$

where the last step is by virtue of (5), since $\|\Delta\psi\|_\infty$ and $\|\nabla\psi\|$ are known to be bounded by Lemmas 5 and 1. The β term can be bounded by using Hölder's inequality as

$$\left|\int_\Omega\beta\nabla\psi_x\cdot\nabla(\Delta\psi)\right|$$
$$\le\beta\|\nabla\psi_x\|\,\|\nabla(\Delta\psi)\|\le\beta C_{10}\|\Delta\psi\|\,\|\nabla(\Delta\psi)\|\le\beta C_{11}\|\Delta\phi\|\,\|\nabla(\Delta\psi)\|,$$

where (4) has been used to obtain the second inequality, and Lemma 1 the last. The next term disappears when integrated by parts, since $\nabla\psi = 0$ on $\partial\Omega$. The final term simplifies to $\int_\Omega\varepsilon[\Delta^2\psi_x\Delta\psi_x + \Delta^2\psi_y\Delta\psi_y] = -\varepsilon\int_\Omega[\Delta^2\psi\Delta\psi_{xx} + \Delta^2\psi\Delta\psi_{yy}] = -\varepsilon\|\Delta^2\psi\|^2$, where the first step is by integrating by parts and using the fact that $\nabla(\Delta\psi)$ must vanish on $\partial\Omega$ since, by Lemma 2, it is integrable. By collecting terms, (11) is obtained. ∎

LEMMA 7. *Let ψ obey* (2) *in Q_T. Then, there exist positive constants C_1, C_2, and C_3 such that for all $t\in[0,T]$,*

$$\frac{1}{2}\frac{d}{dt}\|\Delta^2\psi\|^2 + \varepsilon\|\nabla(\Delta^2\psi)\|^2$$
$$\le\beta C_1\|\nabla(\Delta\psi)\|\,\|\Delta^2\psi\| + \left(C_2 + C_3|\log\|\Delta^2\psi\||\right)\|\Delta^2\psi\|^2. \quad (12)$$

Proof. Apply the Laplacian operator Δ to (2) to get, after some algebra,

$$\Delta^2\psi_t + gj_x\Delta^2\psi_y - \psi_y\Delta^2\psi_x + 2(\psi_{xx} - \psi_{yy})\Delta\psi_{xy} + 2\psi_{xy}(\Delta\psi_{yy} - \Delta\psi_{xx}) + \beta\Delta\psi_x = \varepsilon\Delta^3\psi.$$

We multiply the above by $\Delta^2\psi$, and integrate over Ω. The first term yields that of (12), whereas the next two integrate to zero by parts. Hölder's inequality bounds the term involving β, and

$$\begin{aligned}
&\left|\int_\Omega 2(\psi_{xx} - \psi_{yy})\Delta\psi_{xy}\Delta^2\psi\right| \\
&\quad\le 2\sup_{Q_T}\{|\psi_{xx}| + |\psi_{yy}|\}\left(\int_\Omega(\Delta^2\psi)^2\right)^{1/2}\left(\int_\Omega(\Delta\psi_{xy})^2\right)^{1/2} \\
&\quad\le (C_1 + C_2|\log\|\Delta^2\psi\||)\|\Delta^2\psi\|\left(\int_\Omega(\Delta\psi_{xy})^2\right)^{1/2} \\
&\quad\le (C_3 + C_4|\log\|\Delta^2\psi\||)\|\Delta^2\psi\|^2,
\end{aligned}$$

where the inequalities are obtained from Hölder's inequality, (5), and (4), respectively. Precisely the same sequence of arguments serve to bound the next term.

$$\begin{aligned}
\left|2\psi_{xy}(\Delta\psi_{yy} - \Delta\psi_{xx})\Delta^2\psi\right| &\le 2\sup_{Q_T}\{|\psi_{xy}|\}\|\Delta\psi_{yy} - \Delta\psi_{xx}\|\,\|\Delta^2\psi\| \\
&\le (C_5 + C_6|\log\|\Delta^2\psi\||)\|\Delta^2\psi\|(\|\Delta\psi_{yy}\| + \|\Delta\psi_{xx}\|) \\
&\le (C_7 + C_8|\log\|\Delta^2\psi\||)\|\Delta^2\psi\|^2.
\end{aligned}$$

The viscous term simplifies as $\int_\Omega \varepsilon\Delta^3\psi\Delta^2\psi = -\varepsilon\|\nabla(\Delta^2\psi)\|$, where the boundary terms vanish since they contain the term $\Delta^2\psi$ which, by (2), must be zero on $\partial\Omega$ since the left side of (2) is zero on the boundary. Inequality (12) follows by combining all these estimates. ∎

LEMMA 8. *Let* ψ *obey* (2). *Then, there exist positive constants* C_1 *and* C_2 *such that for any* $t \in [0, T]$, $\|\nabla(\Delta\psi)\| \le C_1$ *and* $\|\Delta^2\psi\| \le C_2$.

Proof. Let $t \in [0, T]$ and define $p(t) = \|\nabla(\Delta\psi)(t)\|$ and $q(t) = \|\Delta^2\psi(t)\|$. Ignoring the positive terms on the left, we obtain from (11) and (12),

$$\frac{dp}{dt} \le \beta C_3 + (C_4 + C_5|\log q|)p \qquad \text{and}$$

$$\frac{dq}{dt} \le \beta C_6 p + (C_7 + C_8|\log q|)q.$$

It suffices to show that any solutions p and q which obey the above evolution inequalities do not blow up in finite time. Note that since both p and q are by definition nonnegative, the only way they can blow up is by going to $+\infty$. Thus, $\|\nabla(\Delta\psi)\|$ and $\|\Delta^2\psi\|$ are bounded by the solutions to the ordinary differential system

$$\left.\begin{aligned} \dot{p} &= \beta C_3 + \left(C_4 + C_5|\log q|\right)p \\ \dot{q} &= \beta C_6 p + \left(C_7 + C_8|\log q|\right)q. \end{aligned}\right\} \tag{13}$$

If p blows up, then so must q, and vice versa. Thus, if there is blow up in finite time for the system (13), then both p and q must blow up. Pick an initial condition (p_0, q_0) to (13) such that $q_0 > e$ and there exist positive constants C_9 and C_{10} such that $\beta C_3 + (C_4 + C_5|\log q|)p \leq C_9 p \log q$ and $\beta C_6 p + (C_7 + C_8|\log q|)q \leq C_{10} q \log q$ for any $p > p_0$ and $q > q_0$. The absolute values on $\log q$ have been discarded since $q_0 > e$ and $\dot{q} > 0$, ensuring that $\log q > 0$ for all time. Thus, the solutions to (13) are bounded by the solutions to the system

$$\left.\begin{aligned} \dot{p} &= C_9 p \log q \\ \dot{q} &= C_{10} q \log q. \end{aligned}\right\} \tag{14}$$

System (14) is easily solved for q, yielding $q(t) = q_0 \exp e^{C_{10} t} \leq q_0 \exp e^{C_{10} T}$, which is uniformly bounded in $[0, T]$ for any finite T. Substituting this in the $\dot{p}$ equation of (14) provides the gross bound $p(t) \leq p_0 \exp\{C_9 T(\log q_0 + e^{C_{10} T})\}$ and thus p and q cannot blow up in finite time. If the initial conditions are chosen to be smaller than p_0 and q_0, and there is blow up in finite time, the solutions must eventually become larger than these values, and therefore this argument holds for all initial conditions. ∎

LEMMA 9. *Let $\psi(x, y, t; \varepsilon)$ solve (2) in Q_T. Then, there exists positive constants C_1, C_2, and C_3 such that for all $t \in [0, T]$,*

$$\sup_{(x,y)\in\Omega} \sum_{|k|=2} \left|D^k\psi(x,y,t)\right| \leq C_1, \qquad \sum_{k=|3|} \left\|D^k\psi(t)\right\| \leq C_2, \qquad \textit{and}$$

$$\sum_{k=|4|} \left\|D^k\psi(t)\right\| \leq C_3.$$

Proof. The first is a simple consequence of Lemma 8 and (5), whereas the last two follow from Lemma 8 and (4). ∎

LEMMA 10. *Let $\psi(x, y, t; \varepsilon)$ solve (2) in Q_T. Then, for any $p \geq 2$, there exist positive constants $C_1(p)$ and $C_2(p)$ such that for all $t \in [0, T]$, $\sum_{|k|=2}\|D^k\psi(t)\|_p \leq C_1(p)$ and $\sum_{|k|=3}\|D^k\psi(t)\|_p \leq C_2(p)$.*

Proof. Each second spatial derivative of ψ is in $W^{1,2}(\Omega)$ for each $t \in [0,T]$, by Lemmas 2, 8, and (4). The limiting version of the Sobolev Embedding Theorem provides the extension to any $L^p(\Omega)$. A similar argument holds for the third spatial derivatives. ∎

LEMMA 11. *Suppose* $\psi(x,y,t;\varepsilon)$ *satisfies* (2) *in* Q_T. *Then, there exists a positive constant* C_3 *such that for any* $t \in [0,T]$, $\sum_{|k|=4} \|D^k\psi(t)\|_4 \le C_3$.

Proof. We apply the Laplacian operator to (2) to obtain, as in Lemma 7,

$$\Delta^2\psi_t + \psi_x\Delta^2\psi_y - \psi_y\Delta^2\psi_x + 2(\psi_{xx} - \psi_{yy})\Delta\psi_{xy} + 2\psi_{xy}(\Delta\psi_{yy} - \Delta\psi_{xx}) + \beta\Delta\psi_x = \varepsilon\Delta^3\psi.$$

We now multiply the above expression by $(\Delta^2\psi)^3$, and integrate over Ω. The first term is simplified as usual, while the next two terms integrate to zero by parts. The next term is bounded by

$$\left|2\int_\Omega (\psi_{xx} - \psi_{yy})\Delta\psi_{xy}(\Delta^2\psi)^3\right| \le 2\sup_{Q_T}\{|\psi_{xx}| + |\psi_{yy}|\}\int_\Omega |\Delta\psi_{xy}|\,|\Delta^2\psi|^3$$
$$= C_4\left\|(\Delta^2\psi)^3\right\|_{4/3}\|\Delta\psi_{xy}\|_4 \le C_5\|\Delta^2\psi\|_4^4,$$

where the inequalities are by virtue of Lemma 9, Hölder's inequality, and (4). A similar sequence of arguments bounds the next two terms by a similar expression, and Hölder's inequality and Lemma 10 bound the β term. By appropriately integrating the viscous term by parts, we obtain

$$\frac{1}{4}\frac{d}{dt}\|\Delta^2\psi\|_4^4 + 3\varepsilon\int_\Omega (\Delta^2\psi)^2\left|\nabla(\Delta^2\psi)\right|^2 \le C_7\|\Delta^2\psi\|_4^3 + C_8\|\Delta^2\psi\|_4^4.$$

By the usual strategy, it is readily seen that $\|\Delta^2\psi(t)\|_4 \le \|\Delta^2\phi\|_4 + (C_7/C_8)(e^{C_8T} - 1)$. The extension to all fourth derivatives is obtained by repeated applications of (4). ∎

LEMMA 12. *Suppose* $\psi(x,y,t;\varepsilon)$ *obeys* (2) *in* Q_T. *Then, there exists a constant* C_5 *such that for all* $t \in [0,T]$, $\sum_{|k|=3} \|D^k\psi(t)\|_{C^0} \le C_5$.

Proof. By Lemma 10 with $p = 4$, we see that any third spatial derivative of ψ is in the Sobolev space $W^{1,4}(\Omega)$ for each such t. Thus, by Sobolev embedding, the C^0 norm of each third spatial derivative of ψ can be bounded by the $W^{1,4}$ norm. ∎

Assuming that the initial condition ϕ satisfies Hypothesis 2, we have obtained a large number of bounds for various norms of the solution to

(2). We summarise these estimates in the following proposition. These bounds can be obtained similarly for (1) by ensuring that the previous arguments remain valid when $\varepsilon = 0$.

PROPOSITION 3. *Let $\psi(x, y, t; \varepsilon)$ satisfy* (2) *in Q_T, and suppose Hypotheses* 1 *and* 2 *are met. Then, there exist positive constants C_1, C_2, C_3, and C_4 depending only upon the norms listed in Hypothesis* 2, *T, β, and, in the case of C_2, p, such that the following hold uniformly for $t \in [0, T]$*:

$$\sum_{1 \le |k| \le 4} \| D^k \psi(t) \| \le C_1, \qquad \sum_{1 \le |k| \le 3} \| D^k \psi(t) \|_p \le C_2(p) \ \forall p \ge 2,$$

$$\sum_{1 \le |k| \le 3} \| D^k \psi(t) \|_{C^0} \le C_3 \qquad \text{and} \qquad \sum_{|k| = 4} \| D^k \psi(t) \|_4 \le C_4.$$

The same estimates hold for $\psi^0(x, y, t)$, which satisfies (1).

4. VANISHING VISCOSITY IN $L^2(\Omega)$ AND $C^1(\Omega)$

Let $\mathbf{u}(x, y, t; \varepsilon)$ be the Eulerian velocity field corresponding to $\psi(x, y, t; \varepsilon)$ which solves (2), and let $\mathbf{u^0}(x, y, t)$ be the Eulerian velocity corresponding to the solution $\psi^0(x, y, t)$ of (1). Then, $\mathbf{u} = J \nabla \psi$ and $\mathbf{u^0} = J \nabla \psi^0$, where J is the symplectic matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is clear that $\|\mathbf{u}\| = \|\nabla \psi\|$ and $\|\mathbf{u^0}\| = \|\nabla \psi^0\|$.

LEMMA 13 (L^2 Vanishing Viscosity). *Suppose Hypotheses* 1 *and* 2 *are met, and that the solutions ψ^0 of* (1) *and ψ of* (2) *both exist in Q_T. Then, there exists a positive constant C_1, depending only on T, β, and the norms of Hypothesis* 2 *such that for any $t \in [0, T]$,*

$$\| \mathbf{u}(t) - \mathbf{u^0}(t) \| \le \varepsilon C_1.$$

Proof. This proof is based on that by Ladyzhenskaya [8]. In Ω, $\psi(x, y, t; \varepsilon)$ satisfies

$$\Delta \psi_t + \psi_x \big(\Delta \psi_y + \beta \big) - \psi_y \Delta \psi_x = \varepsilon \Delta^2 \psi. \tag{15}$$

Similarly, $\psi^0(x, y, t)$ is a solution of (1), i.e.,

$$\Delta \psi_t^0 + \psi_x^0 \big(\Delta \psi_y^0 + \beta \big) - \psi_y^0 \Delta \psi_x^0 = 0. \tag{16}$$

Define $w(x, y, t; \varepsilon) \equiv \psi(x, y, t; \varepsilon) - \psi^0(x, y, t)$. Notice, from the boundary and initial conditions associated with (2) and (1), that $w(\partial \Omega, t; \varepsilon) = 0$ and $w(\Omega, 0; \varepsilon) = 0$. Subtracting (16) from (15), multiplying through by w, and applying the operator $\int_\Omega$ yield

$$\int_\Omega \Big[w\Delta w_t + \beta w w_x + w\big(w_x \Delta\psi_y - w_y \Delta\psi_x\big) + w\big(\psi_x^0 \Delta w_y - \psi_y^0 \Delta w_x\big) - \varepsilon w \Delta^2 \psi \Big] = 0. \tag{17}$$

The resulting terms are handled individually. The first term is handled in the usual fashion, while that involving β is zero since $w = 0$ on $\partial\Omega$. The next term integrates to zero by parts, and the viscous term simplifies to $-\varepsilon\int_\Omega w\Delta^2\psi = -\varepsilon\int_\Omega[w(\Delta\psi_x)_x + w(\Delta\psi_y)_y] = \varepsilon\int_\Omega[w_x\Delta\psi_x + w_y\Delta\psi_y]$ by the usual arguments. Now, the remaining terms of (17) become

$$\int_\Omega \Big[w\psi_x^0 \Delta w_y - w\psi_y^0 \Delta w_x \Big] = \int_\Omega \Big[w_x \psi_y^0 - w_y \psi_x^0 \Big] \Delta w \tag{18}$$

by integrating by parts. Notice that the identity $w_x \Delta w = (1/2)(\partial w_x^2/\partial x) + \partial(w_x w_y)/\partial y - (1/2)(\partial w_y^2/\partial x)$ holds, and analogously for $w_y \Delta w$. Substituting in (18), and integrating by parts again, yields the expression $\int_\Omega[w\psi_x^0\Delta w_y - w\psi_y^0\Delta w_x] = = \int_\Omega[\psi_{xy}^0(w_y^2 - w_x^2) + (\psi_{xx}^0 - \psi_{yy}^0)w_x w_y]$. Substituting everything in (17), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla w(t)\|^2 = -2\varepsilon\int_\Omega \big[w_x \Delta\psi_x + w_y \Delta\psi_y \big] - 2\int_\Omega \Big[\psi_{xy}^0\big(w_y^2 - w_x^2\big) + \big(\psi_{xx}^0 - \psi_{yy}^0\big) w_x w_y \Big]. \tag{19}$$

We now search for estimates for the terms on the right hand side of (19). By Hölder's inequality, and the bound on $\|\nabla(\Delta\psi)\|$ furnished by Lemma 8, notice that the inequality $|2\varepsilon\int_\Omega[w_x\Delta\psi_x + w_y\Delta\psi_y]| \le 2\varepsilon\int_\Omega|\nabla w \cdot \nabla(\Delta\psi)| \le \varepsilon C_1\|\nabla w(t)\|$ holds. Also,

$$\left| \int_\Omega \Big[\psi_{xy}^0\big(w_y^2 - w_x^2\big) + \big(\psi_{xx}^0 - \psi_{yy}^0\big) w_x w_y \Big] \right| \le \int_\Omega \left[|\psi_{xy}^0|\big(w_y^2 + w_x^2\big) + \big(|\psi_{xx}^0| + |\psi_{yy}^0|\big)\left(\frac{w_x^2 + w_y^2}{2}\right) \right].$$

But by Proposition 3 we have the bound $\sup_{Q_T}\{|\psi_{xx}^0| + |\psi_{xy}^0| + |\psi_{yy}^0|\} \le C_2$, and hence the above is bounded by $C_2\|\nabla w(t)\|^2$. Now, from (19), $(1/2)(d/dt)\|\nabla w(t)\|^2 \le \varepsilon C_1\|\nabla w(t)\| + C_2\|\nabla w(t)\|^2$, from which it is easily concluded that, for any $\tau \in [0, T]$, $\|\mathbf{u}(\tau) - \mathbf{u}^0(\tau)\| = \|\nabla w(\tau)\| \le \varepsilon(C_1/C_2)(e^{C_2 T} - 1)$. ∎

THEOREM 1. *Suppose Hypotheses* 1 *and* 2 *are met, and that the solutions* ψ^0 *of* (1) *and* ψ *of* (2) *both exist in* Q_T, *where* $T \in (0, \infty)$. *Then, the solution* $\mathbf{u}$ *of* (2) *tends to the solution* $\mathbf{u}^0$ *of* (1) *as* ε *goes to zero in the* C^1 *norm of* Ω *at each* $t \in [0, T]$; *i.e., for any such* t,

$$\lim_{\varepsilon \downarrow 0} \left(\left\| \mathbf{u}(t) - \mathbf{u^0}(t) \right\|_\infty + \left\| \mathbf{u}_x(t) - \mathbf{u}^\mathbf{0}_x(t) \right\|_\infty + \left\| \mathbf{u}_y(t) - \mathbf{u}^\mathbf{0}_y(t) \right\|_\infty \right) = 0.$$

Proof. This is a modification of the proof by Golovkin [13]. Fix $t \in [0, T]$, and pick $v = \psi(x, y, t; \varepsilon) - \psi^0(x, y, t)$ in the result (6). Thus, for any $\delta > 0$ there exists C_δ such that

$$\begin{aligned} &\left\|(\psi - \psi^0)_x\right\|_\infty + \left\|(\psi - \psi^0)_y\right\|_\infty + \left\|(\psi - \psi^0)_{yx}\right\|_\infty + \left\|(\psi - \psi^0)_{yy}\right\|_\infty \\ &\quad + \left\|(\psi - \psi^0)_{xx}\right\|_\infty + \left\|(\psi - \psi^0)_{xy}\right\|_\infty \\ &\qquad \le \delta\Big(\left\|(\psi - \psi^0)_x\right\| + \left\|(\psi - \psi^0)_y\right\| \left\|\Delta^2(\psi - \psi^0)\right\|\Big) \\ &\qquad\quad + C_8\Big(\left\|(\psi - \psi^0)_x\right\| + \left\|(\psi - \psi^0)_y\right\|\Big), \end{aligned}$$

and therefore

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u^0}\|_\infty + \left\|(\mathbf{u} - \mathbf{u^0})_x\right\|_\infty + \left\|(\mathbf{u} - \mathbf{u^0})_y\right\|_\infty \\ &\qquad \le \delta\Big(\|\mathbf{u} - \mathbf{u^0}\| + \left\|\Delta^2(\psi - \psi^0)\right\|\Big) + C_\delta\|\mathbf{u} - \mathbf{u^0}\| \\ &\qquad \le \delta\Big(\varepsilon C_1 + \left\|\Delta^2(\psi - \psi^0)\right\|\Big) + \varepsilon C_1 C_\delta, \end{aligned}$$

where the second step is by virtue of Lemma 13. Think of δ as a function of ε that tends sufficiently slowly to zero as $\varepsilon \downarrow 0$. We now consider applying the limit $\varepsilon \downarrow 0$ to the above, and note that C_δ may go to infinity as ε (and hence δ) goes to zero. However, if $\delta(\varepsilon)$ is chosen to decay sufficiently slowly as ε goes to zero, it can be ensured that the term $\varepsilon C_1 C_\delta$ goes to zero as $\varepsilon \downarrow 0$. Moreover, the term $\|\Delta^2(\psi - \psi^0)\|$ is known to be finite by Lemma 5 and Proposition 3, and thus the entire right hand side of the above expression tends to zero as ε goes to zero. This proves the result. ∎

5. DECAY RATES AND LIPSCHITZ CONTINUITY IN ε

The result of Theorem 1 is important, yet it fails to provide information on the rate of convergence as $\varepsilon \downarrow 0$. However, without additional assumptions, we are only able to derive such a rate of convergence in the weaker C^0 norm. It turns out that whether $\nabla\psi(x, y, t; \varepsilon)$ is Lipschitz continuous in

ε is closely related to this issue of how the difference of the velocity fields of (1) and (2) decays in ε. Therefore, we first show Lipschitz continuity of $\nabla\psi(x, y, t; \varepsilon)$ in ε, for $\varepsilon \in I \cup \{0\} = [0, \varepsilon_0]$, where ε_0 is sufficiently small. Pick ε_1 and ε_2 in I, and let $\gamma^1(x.y.t)$ and $\gamma^2(x, y, t)$ be the streamfunctions associated with these viscosity values, respectively. From Eq. (2), γ^1 and γ^2 are seen to obey

$$\Delta\gamma_t^1 + \gamma_x^1\Delta\gamma_y^1 - \gamma_y^1\Delta\gamma_x^1 + \beta\gamma_x^1 = \varepsilon_1\Delta^2\gamma^1$$

and

$$\Delta\gamma_t^2 + \gamma_x^2\Delta\gamma_y^2 - \gamma_y^2\Delta\gamma_x^2 + \beta\gamma_x^2 = \varepsilon_2\Delta^2\gamma^2,$$

in Ω, while each being subjected to identical initial and boundary conditions

$$\gamma^1|_{\partial\Omega} = \gamma^2|_{\partial\Omega} = \Delta\gamma^1|_{\partial\Omega} = \Delta\gamma^2|_{\partial\Omega} = 0 \qquad \text{and}$$
$$\gamma^1(x, y, 0) = \gamma^2(x, y, 0) = \phi(x, y).$$

Define $v(x, y, t) = \gamma^1(x, y, t) - \gamma^2(x, y, t)$. By subtracting the γ^2 equation from the γ^1 one, we find that v must satisfy

$$\begin{aligned}\Delta v_t &+ \left(\gamma_x^1\Delta v_y - \gamma_y^1\Delta v_x\right) + \left(v_x\Delta\gamma_y^2 - v_y\Delta\gamma_x^2\right) + \beta v_x \\ &= (\varepsilon_1 - \varepsilon_2)\Delta^2\gamma^2 + \varepsilon_2\Delta^2 v.\end{aligned} \tag{20}$$

LEMMA 14. *There exists a constant C_1 such that for all $t \in [0, T]$, $\|\nabla v(t)\| \le |\varepsilon_1 - \varepsilon_2|C_1$.*

Proof. We multiply (20) by v, and integrate over Ω. The steps of Lemma 13 can be followed more or less directly to obtain the estimate $(1/2)(d/dt)\|\nabla v\|^2 + \varepsilon_2\|\Delta v\|^2 \le C_2\|\nabla v\|^2 + C_3|\varepsilon_1 - \varepsilon_2|\,\|\nabla v\|$. The result follows by proceeding as in Lemma 13. ∎

LEMMA 15. *There exists a constant C_2 such that for all $t \in [0, T]$, $\sum_{|k|=2}\|D^k v(t)\| \le |\varepsilon_1 - \varepsilon_2|C_2$.*

Proof. We multiply (20) by Δv, and integrate over Ω. Following the standard techniques of previous lemmas, and appealing to the results of Lemmas 2, 14, and 8, lead to $(1/2)(d/dt)\|\Delta v\|^2 + \varepsilon_2\|\nabla(\Delta v)\|^2 \le C_5|\varepsilon_1 - \varepsilon_2|\,\|\Delta v\|$. Discarding the positive term from the left, this can be easily integrated to yield $\|\Delta v(t)\| \le C_6|\varepsilon_1 - \varepsilon_2|$ uniformly for $t \in [0, T]$. Inequality (4) grants the necessary extension. ∎

LEMMA 16. *For any $p \ge 2$, there exists a positive constant $C_3(p)$ such that for any $t \in [0, T]$, $\sum_{|k|=1}\|D^k v(t)\|_p \le C_3(p)|\varepsilon_1 - \varepsilon_2|$.*

Proof. The results of Lemmas 14 and 15 indicate that each component of the function $\nabla v(t)/|\varepsilon_1 - \varepsilon_2|$ is in $W^{1,2}(\Omega$, with a bound independent of the εs. The Sobolev Embedding Theorem [20] provides the result. ∎

LEMMA 17. *There exists a constant* C_4 *such that for any* $t \in [0, T]$, $\sum_{|k|=2} \|D^k v(t)\|_4 \leq C_4 |\varepsilon_1 - \varepsilon_2|$.

Proof. We multiply (20) by $(\Delta v)^3$ and integrate over Ω. The first term is handled as usual, while the next terms admit the bound

$$\left| \int_\Omega \left[v_x \Delta \gamma^1_y - v_y \Delta \gamma^1_x \right] (\Delta v)^3 \right| \leq C_1 \sup_{Q_T} \sum_{|k|=3} |D^k \gamma^1| \left[\|v_x\|_4 + \|v_y\|_4 \right] \|\Delta v\|_4^3$$

$$\leq |\varepsilon_1 - \varepsilon_2| C_2 \|\Delta v\|_4^3,$$

where Hölder's inequality, the C^0 bound on the third derivatives of γ^1 given by Proposition 3 and Lemma 16 have been used. A similar bound exists for the β term, whereas integration by parts annihilates the other terms on the left hand side of (20). We also have the bound

$$\left| \int_\Omega (\varepsilon_1 - \varepsilon_2) \Delta^2 \gamma^1 (\Delta v)^3 \right| \leq C_4 \|\Delta v\|_4^3 \|\Delta^2 \gamma^1\|_4 |\varepsilon_1 - \varepsilon_2|$$

$$\leq C_5 |\varepsilon_1 - \varepsilon_2| \, \|\Delta v\|_4^3,$$

where Lemma 11 furnishes the estimate on $\|\Delta^2 \gamma^1\|_4$. The final term can be integrated by parts to yield $\int_\Omega \varepsilon_2 \Delta^2 v (\Delta v)^3 = -3\varepsilon_2 \int_\Omega [(\Delta v_x)^2 + (\Delta v_y)^2](\Delta v)^2$, which is inherently non-positive. Therefore, for each $t \in [0, T]$, $(1/4)(d/dt)\|\Delta v\|_4^4 \leq C_6 |\varepsilon_1 - \varepsilon_2| \, \|\Delta v\|_4^3$. The proof is completed via the standard arguments used before. ∎

THEOREM 2. *For all* $(x, y, t) \in \Omega \times [0, T]$ *and* $\varepsilon \in I \cup \{0\}$, *the function* $\nabla \psi(x, y, t; \varepsilon)$ *satisfying* (2) *is uniformly Lipschitz in* ε.

Proof. Note that $\nabla v(t)/|\varepsilon_1 - \varepsilon_2| = (\nabla \gamma^1(t) - \nabla \gamma^2(t))/|\varepsilon_1 - \varepsilon_2| = (\nabla \psi(t; \varepsilon_1) - \nabla \psi(t; \varepsilon_2))/|\varepsilon_1 - \varepsilon_2|$, where $\psi(x, y, t; \varepsilon)$ satisfies the viscous equation (2), and the (x, y) dependence has been suppressed for convenience. Lemmas 16 and 17 show that the above function is in $W^{1,4}(\Omega)$ uniformly for $t \in [0, T]$. The Sobolev Embedding Theorem permits the extension to $C^0(\Omega)$, and

$$\frac{\| \nabla [\psi(t; \varepsilon_1) - \psi(t; \varepsilon_2)] \|_{C^0(\Omega)}}{|\varepsilon_1 - \varepsilon_2|} \leq C_2.$$

Should $\varepsilon_i = 0$, more care is required since the boundary condition $\Delta \gamma_i = 0$ no longer applies. However, it is readily seen that the previous lemmas of this section hold, since that boundary condition was only necessary to handle the viscous term. Therefore, the Lipschitz condition can be extended to $\varepsilon \in I \cup \{0\} = [0, \varepsilon_0]$. ∎

THEOREM 3. *Suppose $\psi(x, y, t; \varepsilon)$ and $\psi^0(x, y, t)$ satisfy the viscous and inviscid equations* (2) *and* (1), *respectively, and that Hypotheses* 1 *and* 2 *are met. Then, there exists a constant C_1 such that for all $t \in [0, T]$,*

$$\|\mathbf{u}(t) - \mathbf{u^0}(t)\|_{C^0(\Omega)} \leq \varepsilon C_1.$$

Proof. From Theorem 2, we immediately obtain, for any $t \in [0, T]$, the existence of a constant C_1 such that $\|\nabla[\psi(x, y, t; \varepsilon) - \psi^0(x, y, t)]\|_{C^0(\Omega)} \leq C_1 \varepsilon$. The required estimate is obtained by observing that $\mathbf{u} = J\nabla\psi$ and $\mathbf{u^0} = J\nabla\psi^0$. ∎

6. VANISHING VISCOSITY IN $C^3(\Omega)$

The $O(\varepsilon)$ closeness result in $C^0(\Omega)$ for any $t \in [0, T]$, $(x, y) \in \Omega$, and $\varepsilon \in I \cup \{0\}$ permits the expression

$$\nabla\psi(x, y, t; \varepsilon) = \nabla\psi^0(x, y, t) + \varepsilon\nabla\psi^1(x, y, t; \varepsilon), \tag{21}$$

where $\nabla\psi^1(x, y, t; \varepsilon)$ is uniformly bounded in $\Omega \times [0, T] \times [0, \varepsilon_0]$. Taking the divergence of the above expression and adding the quantity βy to each side generates a similar expansion for the barotropic potential vorticity,

$$q(x, y, t; \varepsilon) = q^0(x, y, t) + \varepsilon q^1(x, y, t; \varepsilon), \tag{22}$$

where $q^0 = \Delta\psi^0(x, y, t) + \beta y$ and $q^1(x, y, t; \varepsilon) = \Delta\psi^1(x, y, t; \varepsilon)$. The expansions (21) and (22) may now be substituted in (2), and the fact that ψ^0 satisfies (1) used, to obtain the following IBVP for the perturbation ψ^1:

$$\left.\begin{array}{l} \Delta\psi_t^1 + \{\psi, \Delta\psi^1\} + \{\psi^1, \Delta\psi^0\} + \beta\psi_x^1 = \Delta^2\psi^0 + \varepsilon\Delta^2\psi^1 \quad \text{in } \Omega \\ \psi^1|_{\partial\Omega} = \Delta\psi^1|_{\partial\Omega} = 0 \\ \psi^1|_{t=0} = 0. \end{array}\right\} \tag{23}$$

HYPOTHESIS 3. *The quantities*

$$\sum_{5 \leq |k| \leq 7} \|D^k\psi^0(t)\| \qquad \textit{and} \qquad \sum_{4 \leq |k| \leq 7} \|D^k\psi^0(t)\|_4$$

are bounded independently of $t \in [0, T]$.

Hypotheses 3 is reasonable in that, for $\beta = 0$, it is known that the solution to (1) can be shown to be as smooth as the initial condition [11]. Under these hypotheses, stronger results than those of the previous section concerning the convergence of $\nabla\psi$ to $\nabla\psi^0$ as $\varepsilon \to 0$ can be obtained. The proof strategies are similar but tedious, and we sacrifice detail in the interests of brevity.

LEMMA 18. *$\sum_{|k|=2}\|D^k\psi^1(t)\|$ is uniformly bounded in $[0, T]$.*

Proof. We multiply (23) by $\Delta\psi^1$, and integrate over Ω as usual. The first two terms are simplified as usual, while the next admits the bound

$$\left|\int_\Omega \{\psi^1, \Delta\psi^0\}\Delta\psi^1\right| = \left|\int_\Omega \left[\psi_x^1\Delta\psi_y^0 - \psi_y^1\Delta\psi_x^0\right]\Delta\psi^1\right| \le C_1\left\|\nabla\left(\Delta\psi^0\right)\right\| \|\Delta\psi^1\|$$
$$\le C_2\|\Delta\psi^1\|,$$

because $\nabla\psi^1$ is known to be uniformly bounded in $[0, T]$ by Theorem 3, and $\|\nabla(\Delta\psi^0)\|$ similarly bounded by Proposition 3. The term involving β integrates to zero as in Lemma 2. Since $\Delta^2\psi^0$ has bounded $L^2(\Omega)$ norm, Hölder's inequality shows that $\int_\Omega \Delta^2\psi^0\Delta\psi^1 \le C_3\|\Delta\psi^1\|$. The standard procedure results in $(1/2)(d/dt)\|\Delta\psi^1\|^2 + \varepsilon\|\nabla(\Delta\psi^1)\|^2 \le C_2\|\Delta\psi^1\| + C_3\|\Delta\psi^1\|$, whence, with the help of (4), the required result is obtained. ∎

LEMMA 19. *$\sup_{Q_T}|\Delta\psi^1(x, y, t)|$ is bounded independently of ε.*

Proof. This is based on a maximal principle argument similar to Lemma 5. We write (23) as $q_t^1 + \psi_x q_y^1 - \psi_y q_x^1 + L = \varepsilon\Delta q^1$, where $q^1 = \Delta\psi^1$, and the quantity L consists of terms which have already been shown to be bounded in the supremum norm in Q_T. Noting that $q^1 = 0$ at $t = 0$ by the initial condition, the procedure of Lemma 5 can be followed exactly to yield $\sup_{Q_T}|q^1| \le \sup_{Q_T}|L|$, and the result follows. ∎

LEMMA 20. *$\sum_{|k|=3}\|D^k\psi^1(t)\|$ is bounded, independently of ε if suitably small, uniformly in $[0, T]$.*

Proof. This lemma is proved by first showing that the quantity $\|\nabla(\Delta\psi^1)\|$ is bounded in $[0, T]$. Apply the gradient operator ∇ to (23), take the dot product with $\nabla(\Delta\psi^1)$, and integrate over Ω. There are many terms in this expression, some of which can be bounded by extracting $\sum_{1\le|k|\le 2}\|D^k\psi\|_\infty$, $\sum_{|k|=3}\|D^k\psi^0\|$, and $\sum_{|k|=1}\|D^k\psi^1\|_\infty$ from within the integrals, since they are known to be bounded from previous results. The L^2 norm of all second order spatial derivatives of ψ^1 can be bounded by $\|\Delta\psi^1\|$ by (4), which in turn is bounded by Lemma 18. Using the additional fact that $\|\nabla(\Delta^2\psi^0)\|$ is bounded by Hypothesis 3, we obtain $(1/2)(d/dt)\|\nabla q^1\|^2 + \varepsilon\int_\Omega[(q_{xx}^1)^2 + 2(q_{xy}^1)^2 + (q_{yy}^1)^2] \le C_1\|\nabla q^1\|^2 + C_2\|\nabla q^1\|$, from which a bound on all third derivatives of ψ^1 can be derived via the usual arguments. ∎

LEMMA 21. *$\sum_{|k|=4}\|D^k\psi^1(t)\|$ and $\sum_{|k|=2}\|D^k\psi^1(t)\|_{C^0}$ are bounded, independently of ε, uniformly in $[0,T]$.*

Proof. We apply the Laplacian operator to (23) to obtain

$$\begin{aligned}\Delta q_t^1 &+ \left[\Delta\psi_x q_y^1 - \Delta\psi_y q_x^1\right] + 2q_{xy}^1\left[\psi_{xx} - \psi_{yy}\right] + \left[\psi_x \Delta q_y^1 - \psi_y \Delta q_x^1\right] \\ &+ 2\psi_{xy}\left[q_{yy}^1 - q_{xx}^1\right] + \left[q_x^1 \Delta\psi_y^0 - q_y^1 \Delta\psi_x^0\right] + 2\Delta\psi_{xy}^0\left[\psi_{xx}^1 - \psi_{yy}^1\right] \\ &+ \beta q_x^1 + \left[\psi_x^1\Delta^2\psi_y^0 - \psi_y^1\Delta^2\psi_x^0\right] - 2\psi_{xy}^1\left[\Delta\psi_{xx}^0 - \Delta\psi_{yy}^0\right] \\ &= \Delta^3\psi^0 + \varepsilon\Delta^2 q^1. \end{aligned} \tag{24}$$

We now multiply (24) by $\Delta^2\psi^1 = \Delta q^1$, and integrate over Ω as usual. The quantities $\sum_{2\le|k|\le3}|D^k\psi|$, $\sum_{|k|=3}|D^k\psi^0|$, and $\sum_{1\le|k|\le2}|D^k\psi^1|$ are known to be bounded, and can be extracted from the integrals. Moreover, $\sum_{|k|=2}|D^k\psi^1|$ can be estimated by an expression of the form $C_4 + C_5\log\|\Delta^2\psi^1\|$ from (5), and the L^2 norms of general second order derivatives are estimated by the norms of the Laplacian by virtue of (4). Finally, we note that $\|\Delta^3\psi^0\|$ is bounded by Hypothesis 3, and the inequality

$$\frac{1}{2}\frac{d}{dt}\|\Delta q^1\|^2 + \varepsilon\left\|\nabla\left(\Delta q^1\right)\right\|^2 \le C_1\|\Delta q^1\|^2 + C_2\|\Delta q^1\| + C_3\|\Delta q^1\|\log\|\Delta q^1\|$$

results. Disregarding the positive term on the left, this leads to $(d/dt)\|\Delta q^1\| \le C_1\|\Delta q^1\| + C_2 + C_3\log\|\Delta q^1\|$, from which it is clear that $\|\Delta q^1\|$ cannot blow up in finite time, since the growth of the derivative is at most linear. Inequality (4) provides an extension of this result to all second order spatial derivatives of q^1, and the first claim is proved. Now consider the function $\sum_{|k|=2}D^k\psi^1(t)$. By the results of Lemmas 18 and 20, this is in the Sobolev space $W^{2,2}(\Omega)$ for any $t\in[0,T]$. Hence, by the Sobolev Embedding Theorem, this can be bounded in the supremum norm of Ω, and the second claim is proved. ∎

LEMMA 22. *For any $p \ge 2$, the norm $\sum_{|k|=3}\|D^k\psi^1\|_p$ is uniformly bounded in $[0,T]$.*

Proof. From Lemma 21 and (4), it is known that $\sum_{|k|=3}\|D^k\psi^1\|$ and $\sum_{|k|=4}\|D^k\psi^1\|$ are bounded in $[0,T]$. The Sobolev Embedding Theorem furnishes the extension. ∎

LEMMA 23. *$\sum_{|k|=4}\|D^k\psi^1\|_4$ and $\sum_{|k|=3}\|D^k\psi^1\|_{C^0(\Omega)}$ are bounded uniformly for $t\in[0,T]$.*

Proof. Equation (24), obtained by applying the Laplacian to (23), is multiplied by $(\Delta q^1)^3$ and then integrated over Ω. The strategy of Lemma 7 is then used to bound the terms. That $\sum_{|k|=3}\|D^k\psi^1\|_4$ is bounded by

Lemma 22, and that $\|\Delta^2\psi^0\|_4$, $\sum_{|k|=1}\|D^k\Delta^2\psi^0\|_4$ and $\|\Delta^3\psi^0\|_4$ are bounded by Hypothesis 3, prove useful in deriving

$$\frac{1}{4}\frac{d}{dt}\|\Delta^2\psi^1\|_4^4 + \frac{3}{2}\varepsilon\left\|\nabla\left[\left(\Delta^2\psi^1\right)^2\right]\right\| \le \|\Delta^2\psi^1\|_4^3\left[C_1 + C_2\|\Delta^2\psi^1\|_4\right].$$

The standard procedure, with repeated applications of (4), provides a bound on all fourth derivatives of ψ^1. Moreover, this bound along with Lemma 22 ensures that any third spatial derivative of ψ^1 is in $W^{1,4}(\Omega)$, and so by the Sobolev Embedding Theorem, it is in $C^0(\Omega)$. ∎

LEMMA 24. *$\sum_{|k|=5}\|D^k\psi^1\|_4$ and $\sum_{|k|=4}\|D^k\psi^1\|_{C^0}$ are bounded independently of ε in $[0,T]$.*

Proof. As in the proof of Lemma 6, we consider the gradient of (24), then take its dot product with the vector $(\Delta q_x^1, \Delta q_y^1)$ and integrate over Ω. It is apparent that there will be many terms which result from this procedure. Define $\|w(t)\|_* = (\|\Delta q_x^1(t)\|_4^4 + \|\Delta q_y^1(t)\|_4^4)^{1/4}$. It is clear that the $L^4(\Omega)$ norms of Δq_x^1 and Δq_y^1 are bounded by $\|w\|_*$. Quantities known to be bounded from previous lemmas, such as $\sum_{2\le|k|\le 3}|D^k\psi|$, $\sum_{2\le|k|\le 3}|D^k\psi^1|$ and $\sum_{1\le|k|\le 3}|D^k\psi^0|$, can be extracted from the integrals, and Hölder's inequality can be used as in the proof of Lemma 11 to obtain $(d/dt)\|w\|_* \le C_1\|w\|_* + C_2$, from which it is clear that $\|w\|_*$ is bounded for all finite t, and therefore so are $\|\Delta q_x^1\|_4$ and $\|\Delta q_y^1\|_4$. The extension to all fifth spatial derivatives of ψ^1 is clear from two applications of (4). Now, from this result and that of Lemma 23, the function $\sum_{|k|=4}D^k\psi^1(t)$ is in $W^{1,4}(\Omega)$ for each $t\in[0,T]$, and the C^0 bound is obtained from the Sobolev Embedding Theorem. ∎

THEOREM 4. *Let ψ^0 satisfy* (1), *and $\psi = \psi^0 + \varepsilon\psi^1$ satisfy* (23) *in Q_T, and suppose that Hypotheses* 1, 2, *and* 3 *are all met. Then, there exists a constant C_1 such that*

$$\sup_{t\in[0,T]}\left\|\{\mathbf{u}(t) - \mathbf{u^0}(t)\}\right\|_{C^3(\Omega)} \le \varepsilon C_1.$$

Proof. We have shown in Lemmas 21, 23, and 24 that the second, third, and fourth spatial derivatives, respectively, of ψ^1 are bounded in $[0,T]$, uniformly in ε. However, it has been proven in Theorem 3 that a similar bound exists for the first spatial derivative of ψ^1. The observation that $\mathbf{u} - \mathbf{u^0} = J(\nabla\psi - \nabla\psi^0) = \varepsilon J\nabla\psi^1$, where J is the symplectic matrix introduced in Section 4, completes the proof. ∎

Remark 1. The proofs can be modified trivially to show that the same results follow if the gradients of the initial conditions of (1) and (2) are not necessarily equal, but are $O(\varepsilon)$ apart in $C^3(\Omega)$, $W^{3,2}(\Omega)$, and $W^{4,4}(\Omega)$.

Remark 2. The lack of boundaries is essential, since if a physical boundary exists, (2) must satisfy a no-slip condition whereas (1) need not.

Remark 3. The vanishing viscosity results only hold for finite times, since the solution to (2) dissipates to zero at $t = \infty$. This is reflected by the fact that the constants C_i depend on T.

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REFERENCES

1. J. Pedlosky, "Geophysical Fluid Dynamics," Springer-Verlag, New York, 1979.
2. B. Cushman-Roisin, "Introduction to Geophysical Fluid Dynamics," Prentice Hall International, Englewood Cliffs, NJ, 1994.
3. H. S. Balasuriya, "Viscosity Induced Transport in Barotropic Jets," Ph.D. thesis, Brown University, 1996.
4. H. S. Balasuriya and C. K. R. T. Jones, Viscosity induced advection in barotropic jets, in preparation.
5. J. Leray, Essai sur les mouvements plans d'un liquide visqueux que limitent des parois, *J. Math. Pures Appl.* **13** (1934), 331–418.
6. J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.* **63** (1934), 193–248.
7. E. Hopf, Über die Aufgangswertaufgabe für die hydrodynamischen Grundliechungen, *Math. Nachr.* **4** (1951), 213–231.
8. O. A. Ladyzhenskaya, "The Mathematical Theory of Viscous Incompressible Flow," Gordon & Breach, New York, 1969.
9. J. L. Lions and G. Prodi, Un théorème d'existence et d'unicité dans les équations de Navier-Stokes en dimension 2, *C. R. Acad. Sci. Paris* **248** (1959), 3519–3521.
10. J. Serrin, The initial value problem for the Navier-Stokes equations, *in* "Nonlinear Problems" (R. E. Langer, Ed.), pp. 69–98, Univ. of Wisconsin Press, Madison, 1963.
11. D. G. Ebin and J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math. (2)* **92** (1970), 102–163.
12. F. J. McGrath, Nonstationary plane flow of viscous and ideal fluids, *Arch. Rational Mech. Anal.* **27** (1968), 329–348.
13. H. Golovkin, Vanishing viscosity in Cauchy's problem for hydrodynamics equations, *Mem. Steklov Inst.* **92** (1966), 33–53.
14. R. Teman, "Navier-Stokes Equations and Nonlinear Functional Analysis," F Regional Conf. Ser. in Appl. Math., SIAM, Philadelphia, 1983.
15. P. Constantine and C. Foias, "Navier-Stokes Equations," Univ. of Chicago Press, Chicago, 1988.

16. T. Kato and G. Ponce, Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces $L_s^p(\mathbb{R}^2)$, *Rev. Mat. Iberoamericana* **2** (1986), 73–88.
17. A. J. Bourgeois and J. T. Beale, Validity of the quasigeostrophic model for large-scale flow in the atmosphere and ocean, in preparation.
18. P. Constantin and J. Wu, Inviscid limit for vortex patches, *Nonlinearity* **8** (1995), 735–742.
19. P. Constantin and J. Wu, The inviscid limit for non-smooth vorticity, in preparation.
20. R. A. Adams, "Sobolev Spaces," Academic Press, San Diego, 1978.