Nonlinearity **11** (1998) 47–77. Printed in the UK PII: S0951-7715(98)80914-6

Viscous perturbations of vorticity-conserving flows and separatrix splitting

Sanjeeva Balasuriya, Christopher K R T Jones and Björn Sandstede
Division of Applied Mathematics, Brown University, Providence, RI 02912, USA

Received 7 January 1997, in final form 20 August 1997
Recommended by P Constantin

Abstract. We examine the effect of the breaking of vorticity conservation by viscous dissipation on transport in the underlying fluid flow. The transport of interest is between regimes of different characteristic motion and is afforded by the splitting of separatrices. A base flow that is vorticity conserving is therefore assumed to have a separatrix that is either a homoclinic or heteroclinic orbit. The corresponding vorticity dissipating flow, with small time-dependent forcing and viscous parameter ε, maintains an $O(\varepsilon)$ closeness to the inviscid flow in a weak sense. An appropriate Melnikov theory that allows for such weak perturbations is then developed. A surprisingly simple expression for the leading-order distance between perturbed invariant (stable and unstable) manifolds is derived which depends only on the inviscid flow. Finally, the implications for transport in barotropic jets are discussed.

AMS classification scheme numbers: 76C05, 34C37, 76F20, 86A05

1. Introduction

Lagrangian trajectories of fluid parcels in a two-dimensional incompressible fluid are obtained by solving the ordinary differential equation (ODE)

$$\begin{aligned}\dot{x} &= u(x, y, t) \\ \dot{y} &= v(x, y, t)\end{aligned} \tag{1.1}$$

where (u, v) is the Eulerian velocity field which can be expressed in terms of a streamfunction $\psi(x, y, t)$ as $u = -\frac{\partial\psi}{\partial y}$ and $v = \frac{\partial\psi}{\partial x}$. Separatrices consist of distinguished Lagrangian trajectories that demarcate the boundary between regimes of different characteristic motion in a fluid flow. If the velocity field (u, v) is steady (i.e. independent of time) separatrices are formed by homoclinic or heteroclinic orbits of (1.1), see figure 1(a). If, on the other hand, the velocity field is varying in time and only nearly steady, then the homoclinic, or heteroclinic, orbits that formed the separatrices in the steady limit may break. Such a breaking will augur the transport of fluid between regimes of ostensibly different motion. When the separatrix is intact (the steady case) it serves as an impermeable boundary to fluid parcels and therefore genuinely separates the different regimes. When the separatrix has split (the time-varying, near-steady case) fluid parcels can move between these previously distinct regions, see figure 1(b). If the splitting of a heteroclinic cycle or a homoclinic orbit occurs as a transverse intersection of the stable and unstable manifolds involved in the separatrix then the transport will have a chaotic signature and stirring of the fluid is facilitated, see Ottino [21].

0951-7715/98/010047+31$19.50

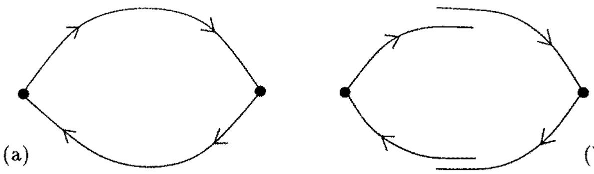

Figure 1. Separatrices formed by a heteroclinic cycle (or cat's eye) (a) before and (b) after perturbation.

A standard approach in the study of this kind of Lagrangian transport in unsteady flows is to take a steady velocity field, (u, v) in (1.1), independent of time, and add a time-dependent perturbation. It is, however, not realistic to consider only cases in which the perturbing unsteady term has an explicit expression. Indeed, the velocity field is found by solving a partial differential equation (PDE), the vorticity equation for two-dimensional incompressible flow, and it is to this equation that the perturbing terms should be added. These additional terms would represent physical effects that need to be taken into account; in this paper we focus on the effects of viscosity and forcing. When these terms are added to the vorticity equation the resulting velocity field found by solving that equation will have been perturbed. However, there is no reason to expect that we would get an explicit expression for the perturbed velocity field, even if the unperturbed field happened to be given in closed form.

We then consider the situation in which an unforced, inviscid velocity field is steady, at least in a moving frame, and has regions protected by separatrices. If viscosity and forcing are added to the system, we anticipate an unsteady velocity field resulting and we ask the question of if, and how, the separatrices split.

To set the scene, let us assume that the streamfunction, the existence of which is guaranteed by incompressibility, is denoted in the inviscid case by $\psi^0(x, y, t)$. The dynamics obeys, to a first approximation, the conservation of vorticity equation

$$\frac{\mathrm{D}q^0}{\mathrm{D}t} = 0 \tag{1.2}$$

where the operator $\frac{\mathrm{D}}{\mathrm{D}t}$ represents the material derivative $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} - \frac{\partial \psi^0}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi^0}{\partial x}\frac{\partial}{\partial y}$, and the vorticity is given by

$$q^0(x, y, t) = \Delta\psi^0(x, y, t) \tag{1.3}$$

where Δ is the Laplacian in the spatial variables. Note that (1.2) can be considered as a nonlinear PDE for the streamfunction ψ^0 alone:

$$\frac{\partial}{\partial t}\Delta\psi^0 - \frac{\partial \psi^0}{\partial y}\frac{\partial \Delta\psi^0}{\partial x} + \frac{\partial \psi^0}{\partial x}\frac{\partial \Delta\psi^0}{\partial y} = 0.$$

The Lagrangian trajectories of fluid parcels are then obtained by solving the ODE

$$\begin{aligned} \dot{x} &= -\frac{\partial \psi^0}{\partial y}(x, y, t) \\ \dot{y} &= \frac{\partial \psi^0}{\partial x}(x, y, t). \end{aligned} \tag{1.4}$$

We add viscosity and forcing to the system. Denoting the streamfunction by $\psi(x, y, t)$, (1.2) is replaced by

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \varepsilon[\Delta q + f(x, y, t)] \tag{1.5}$$

where $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}$, and the vorticity and streamfunction are again related by

$$q(x, y, t) = \Delta\psi(x, y, t). \tag{1.6}$$

The positive parameter ε represents a measure of the viscosity. The corresponding PDE for ψ reads

$$\frac{\partial}{\partial t}\Delta\psi - \frac{\partial \psi}{\partial y}\frac{\partial \Delta\psi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \Delta\psi}{\partial y} = \varepsilon[\Delta^2\psi + f(x, y, t)].$$

Of interest then is the Lagrangian dynamics associated with (1.5), that is the trajectories of

$$\begin{aligned} \dot{x} &= -\frac{\partial \psi}{\partial y}(x, y, t) \\ \dot{y} &= \frac{\partial \psi}{\partial x}(x, y, t). \end{aligned} \tag{1.7}$$

We consider (1.7) as a perturbation of (1.4). It is an instructive exercise to consider the naive application of Melnikov theory which would involve assuming the differentiability of the streamfunction in ϵ. The Melnikov theory would, in principle, allow us to decide whether a separatrix $\bar{z}(t) = (\bar{x}(t), \bar{y}(t))$ of (1.4) breaks under the perturbation. The Melnikov integral would be given by

$$\int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x}\psi^0(\bar{z}(t))\frac{\partial}{\partial y}\frac{\partial}{\partial \varepsilon}\psi(\bar{z}(t), t+\tau)\bigg|_{\varepsilon=0} - \frac{\partial}{\partial y}\psi^0(\bar{z}(t))\frac{\partial}{\partial x}\frac{\partial}{\partial \varepsilon}\psi(\bar{z}(t), t+\tau)\bigg|_{\varepsilon=0} \right) \mathrm{d}t \tag{1.8}$$

see corollary 1 and lemma 1 applied for $Q^0 = \psi^0$. Note however that we only know of the full streamfunction ψ and the inviscid streamfunction ψ^0 that they satisfy their respective PDEs. Therefore, (1.8) is not very helpful even if ψ were differentiable in ε except when $\frac{\partial}{\partial \varepsilon}\psi$ is known explicitly.

Actually, the perturbed streamfunction is in general *not* differentiable with respect to ε, see [16]. Indeed, a fundamental difficulty is that the limiting behaviour of ψ as $\varepsilon \to 0$ can only be established in a weak sense. Even though Ladyzhenskaya [16] proved an estimate

$$\|\psi^0 - \psi\|_{L^2} \leqslant C\varepsilon$$

on compact time intervals, provided ψ^0 is smooth enough, ψ is not differentiable in ε at $\varepsilon = 0$. This is an inevitable difficulty in problems of vanishing dissipation, see, for instance, [30]. For our purposes the consequence is that expression (1.8) is formal. It therefore necessitates an adapted Melnikov theory which works in cases where the perturbation is only weakly related to the limiting flow. A further complication is that we cannot guarantee the existence of a perturbed streamfunction, in other words a solution of (1.5), which is close to ψ^0 for all time, even if the initial data are close. Since we are interested in the behaviour of the associated dynamical system and, in particular, its potentially chaotic nature it is natural to consider velocity fields for the perturbed system that are periodic. The existence, however, cannot be guaranteed of such velocity fields and in section 7 it is shown that periodic velocity fields are indeed unlikely to occur. We therefore choose to develop the theory for the case of bounded velocity fields, and this is the subject of theorem 1.

The key computation, and indeed the main result of this paper, is then to calculate the distance between stable and unstable manifolds in the ODE phase space after separation due to a viscous perturbation. An explicit expression is derived for the leading-order term of this distance. Surprisingly, and in contrast to what might be expected of (1.8), it depends only on the unperturbed streamfunction ψ^0, that is the inviscid fluid, and the forcing term,

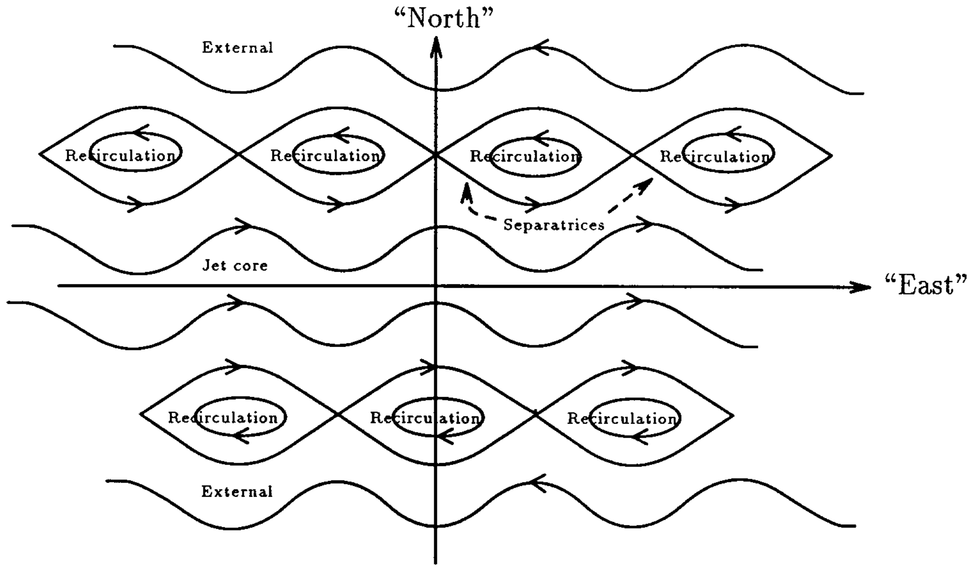

Figure 2. A typical meandering (cat's eye) jet.

see theorem 3. It is then possible to draw conclusions about the nature of transport after adding viscosity from the knowledge of the inviscid velocity field alone.

Much of the motivation for these results comes from oceanography. The relevance of oceanic jets such as the Gulf Stream to fluid transport in the oceans has evoked much recent interest among oceanographers [3–6, 11, 14, 18–20, 23–27, 31]. Under some approximations, these jets can be modelled by *barotropic* motion: a reduction to the two horizontal directions [10, 22]. Satellite photographs show that the Gulf Stream is, close to continental America, an eastward flowing meandering jet, flanked by recirculating regions called *cat's eyes* [27]. A typical (gross) flow pattern of such a jet is illustrated in figure 2, whose axes loosely correspond to the local eastward and northward directions. In fact, it is traditional to assume that a phase portrait of the form of figure 2 arises in a frame moving eastward at some speed c [11, 20, 23, 24, 26, 31]. However, this apparent regularity of motion is challenged by the observed motion of floats, which traverse seemingly random trajectories near the Gulf Stream [5, 6]. The indications are that the Gulf Stream can reasonably be modelled by a regular Eulerian flow which, nevertheless, has irregular Lagrangian motion.

We expect that perturbations will destroy the heteroclinic separatrices of figure 2, producing interaction between fluid parcels of disparate origins. Many authors have exploited this fact in *kinematic models* to obtain, numerically and otherwise, chaotic mixing [3, 14, 19, 26, 31]. However, these perturbations are often imposed without regard to the dynamical equations that the velocity field must obey. Also, streamfunctions are often used which satisfy the vorticity equation only *approximately*, see, for instance, [11, 24]. Equation (1.2) is linearized about an appropriate jet. A superposition of eigenfunctions of the linearized operator is then added to the jet solution. Finally, the resulting function is used as a velocity field in the ODE (1.7) describing Lagrangian trajectories, and chaotic mixing may be found, see [11, 23, 24]. Note that vorticity is, however, not exactly conserved since (1.2) is linearized. Here, we introduce a dynamically consistent approach in that only velocity fields are considered which satisfy either (1.2) or (1.5), and hence either conserve vorticity or dissipate it in a planned and predictable fashion.

An apparent key to the dynamics of the ocean is the near conservation of the *potential vorticity*, which generalizes the notion of vorticity to the oceanographic context by including the effect of planetary vorticity. The *barotropic β-plane potential vorticity* is then given by adding a linear term in y to the ambient vorticity

$$q^0(x, y, t) = \Delta\psi^0(x, y, t) + \beta y. \tag{1.9}$$

The positive constant β is the Coriolis parameter. The set-up described above can be re-interpreted in the oceanographic context by replacing (1.3) with (1.9). Under this re-interpretation, and for the case of a meandering jet, the phase portrait of (1.4) in a frame moving eastward at speed c is then assumed to have the structure of figure 2, and particular examples are given in [1, 7, 11, 24].

As pointed out by Brown and Samelson [7], the conservation of (potential) vorticity exerts significant restrictions on the Lagrangian trajectories of the velocity field. Indeed, it adds a second integral to the Hamiltonian system (1.1) and if the velocity field is periodic, and the vorticity and Hamiltonian are functionally independent then the system is integrable and the kind of Lagrangian transport afforded by broken separatrices cannot occur. Taking a steady velocity field, in a moving frame, for the inviscid, unforced limit is designed to capture the dynamics enforced by the conservation of potential vorticity.

The issue of oceanographic interest is then to see if a velocity field resulting from a situation under which potential vorticity is not conserved does indeed involve transport between the jet, the cat's eyes and the ambient water. This will be addressed exactly as above but with the potential vorticity replacing the usual two-dimensional vorticity. Since the non-oceanographic case is achieved by just setting $\beta = 0$, in the following we shall cast all the results in oceanographic terms and refer to the potential vorticity.

This paper is organized as follows. In section 2, we develop the Melnikov theory for weak perturbations. Estimates for the distances of inner separatrices of perturbed cat's eyes are derived in section 3. Section 4 deals with the validity of considering the Eulerian velocity field resulting from the viscous dynamics (1.5) as a regular perturbation on that produced by the inviscid limit (1.2). We combine these results in sections 5 and 6, where we compute the distance between manifolds in the phase space after separation due to a viscous perturbation. In section 7, we return to the Eulerian equations, and comment on whether periodic streamfunctions occur. Finally, the implications on transport in barotropic jets are discussed in section 8.

2. Melnikov theory for weak perturbations

This section presents a Melnikov theory for base flows in two dimensions which possess heteroclinic structures. The point here is that the perturbations are not necessarily continuous in ε. We also allow for non-periodic time dependencies. The approach is motivated by that presented in [8, section 11.3] for the smooth case. Suppose that Ω is a two-dimensional smooth surface, and $u \in \Omega$. Let $g^0 : \Omega \to \mathbb{R}^2$ such that $g^0 \in C^r(\Omega)$, $r \geqslant 2$. Consider as the unperturbed flow on Ω the autonomous ODE

$$\dot{u} = g^0(u). \tag{2.1}$$

First, we assume the presence of a heteroclinic orbit in the unperturbed system (2.1).

Hypothesis 1. *There exist hyperbolic equilibria A_0 and B_0 of (2.1) with one-dimensional stable and unstable manifolds. A branch of the stable manifold of B_0 (denoted $W^S_{B_0}$) coincides with a branch of the unstable manifold of A_0 (denoted $W^U_{A_0}$). This heteroclinic orbit is denoted by $\bar{u}(t)$.*

Let ∇ be the gradient operator with respect to the two-dimensional variable on Ω such that ∇g^0 is the Jacobian matrix of any function $g^0 : \Omega \to \mathbb{R}^2$.

As a consequence of hypothesis 1, the adjoint variational equation

$$\dot{u} = -\nabla g^0(\bar{u}(t))^* u \tag{2.2}$$

along the heteroclinic orbit $\bar{u}(t)$ possesses a unique, up to a constant multiple, bounded non-zero solution $\varphi(t)$. If, for instance, (2.1) possesses a first integral $Q^0(u)$, the solution $\varphi(t)$ is readily computed.

Lemma 1 ([13]). *If hypothesis 1 is met and (2.1) possesses a first integral $Q^0(u)$, that is, $\frac{\mathrm{d}}{\mathrm{d}t} Q^0(u(t)) = 0$ for any solution $u(t)$ of (2.1), then $\varphi(t) = \nabla Q^0(\bar{u}(t))$ satisfies (2.2).*

Let ε be a parameter in the interval $\mathcal{I} = [0, \varepsilon_0]$, where ε_0 is a positive number assumed to be as small as required. We now consider the perturbed equation

$$\dot{u} = g^0(u) + g^1(u, t, \varepsilon) \tag{2.3}$$

where the function g^1 satisfies the following hypothesis. The operator ∇, as before, will pertain only to the spatial variable u.

Hypothesis 2. *$g^1 : \Omega \times \mathbb{R} \times \mathcal{I} \to \mathbb{R}^2$ satisfies the conditions:*

(i) $g^1 \in C^r(\Omega \times \mathbb{R})$ for each $\varepsilon \in \mathcal{I}$ with uniform bounds, where $r \geqslant 2$;

(ii) $g^1(u, t, 0) = 0$ for all $(u, t) \in \Omega \times \mathbb{R}$; and

(iii) there is a positive constant C such that

$$|g^1(u, t, \varepsilon)| + |\nabla g^1(u, t, \varepsilon)| \leqslant C|\varepsilon|$$

holds uniformly in $(u, t) \in \Omega \times \mathbb{R}$.

Note that we deliberately do not assume any smoothness in ε as we cannot guarantee such smoothness in the application to fluid flow. Condition (iii) is a form of Lipschitz continuity at $\varepsilon = 0$. On the other hand, if hypothesis 2 is satisfied, implicit function theorems are applicable since the perturbation is smooth in the spatial variable and its Jacobian is small.

Under such a perturbation, the hyperbolic equilibrium A_0 perturbs to a bounded solution $A_\varepsilon(t)$. Its stable and unstable manifolds persist for small enough ε, since g^1 is uniformly bounded by hypothesis 2, and similarly for B_0. The proof of this persistence is provided via exponential dichotomies by the roughness theorem of Coppel [9]. The intention now is to develop a *distance function* $d(\tau, \varepsilon)$ which measures the separation between the unstable manifold of $A_\varepsilon(t)$ and the stable manifold of $B_\varepsilon(t)$ in the time slice $\{t = \tau\}$. We begin by defining the space

$$B(\mathbb{R}) = \{G : \mathbb{R} \to \mathbb{R}^2 \text{ bounded and continuous}\}$$

with the norm $|G| = \sup_{t\in\mathbb{R}} |G(t)|$.

Thus, $\varphi(t) \in B(\mathbb{R})$, and moreover decays to zero exponentially as $t \to \pm\infty$. Define the continuous projection operator P on $B(\mathbb{R})$ by

$$PG = \frac{1}{\int_{-\infty}^{\infty} |\varphi(s)|^2 \, \mathrm{d}s} \varphi(t) \int_{-\infty}^{\infty} \varphi(s) \cdot G(s) \, \mathrm{d}s.$$

The following lemma, which is essentially a Lyapunov–Schmidt reduction, now holds.

Lemma 2. *If $G \in B(\mathbb{R})$, the equation*

$$\dot{u} = \nabla g^0(\bar{u}(t))u + G(t) \tag{2.4}$$

has a solution in $B(\mathbb{R})$ if and only if $PG = 0$. If the initial condition $u(0)$ of (2.4) is such that $\langle u(0), g^0(\bar{u}(0))\rangle = 0$, then the solution is unique. Moreover, the solution operator $Q : (\mathrm{id} - P)B(\mathbb{R}) \to B(\mathbb{R})$ is linear and continuous.

Proof. See lemma 3.2 in section 11.3 of [8]. □

The result of lemma 2 can be used to provide a mathematical characterization for the existence of a heteroclinic point of (2.3) near the unperturbed manifold. Let $u(t)$ satisfy (2.3), and set

$$u(t) = \bar{u}(t-\tau) + \xi(t-\tau).$$

The idea is to find a solution $u(t)$ which remains close to the (unperturbed) heteroclinic orbit $\bar{u}(t-\tau)$. Thus, a small solution $\xi(t)$ is sought which must satisfy

$$\begin{aligned}\dot{\xi} &= \nabla g^0(\bar{u}(t))\xi + g^0(\bar{u}(t)+\xi) - g^0(\bar{u}(t)) - \nabla g^0(\bar{u}(t))\xi + g^1(\bar{u}(t)+\xi, t+\tau, \varepsilon)\\ &=: \nabla g^0(\bar{u}(t))\xi + G(\xi, t+\tau, \varepsilon) \end{aligned} \tag{2.5}$$

where the above serves as a definition for the function $G(\xi, t, \varepsilon)$. The existence of a heteroclinic point of (2.3) near the heteroclinic orbit of the unperturbed case depends on the existence of a bounded solution to (2.5), see [8]. By lemma 2, this problem is equivalent to solving the pair of equations

$$PG(\xi, \cdot + \tau, \varepsilon) = 0 \tag{2.6}$$

$$\xi = Q(\mathrm{id} - P)G(\xi, \cdot + \tau, \varepsilon). \tag{2.7}$$

We now state the main theorem which gives a characterization of the existence of a (transverse) heteroclinic point in terms of a Melnikov-type function. Recall that $\varphi(t)$ is the unique bounded solution of the adjoint equation (2.2).

Theorem 1. *Suppose hypothesis 1 holds for the unperturbed flow (2.1), and that the perturbation $g^1(u, t, \varepsilon)$ satisfies hypothesis 2. Then, there exists a unique solution $\bar{\xi}(\tau, \varepsilon)(t)$ of (2.7) for small enough ε. Furthermore,*

$$|\bar{\xi}(\tau, \varepsilon)| \leqslant C|\varepsilon|$$

for some positive constant C uniformly in τ. Define the distance function

$$d(\tau, \varepsilon) = \int_{-\infty}^{\infty} \langle \varphi(t), G(\bar{\xi}(\tau, \varepsilon)(t), t+\tau, \varepsilon)\rangle \, \mathrm{d}t \tag{2.8}$$

then there exists a heteroclinic point of (2.3) in a neighbourhood of $W^U_{A_0} = W^S_{B_0}$ for $|\varepsilon| < \varepsilon_0$ if and only if ε and τ satisfy $d(\tau, \varepsilon) = 0$. Moreover, the intersection is transverse if and only if $\frac{\partial}{\partial \tau} d(\tau, \varepsilon) \neq 0$.

In other words, the unique solution $\bar{\xi}(\tau, \varepsilon)(t)$ of (2.7) satisfies (2.6) if and only if $d(\tau, \varepsilon) = 0$.

Proof. We fix $\tau \in \mathbb{R}$. By hypothesis 2, the Jacobian ∇g^1 is small for ε small. Thus, the operator

$$T(\xi, \varepsilon) = Q(\mathrm{id} - P)G(\xi, \cdot + \tau, \varepsilon)$$

which consists of the sum of a quadratic term in ξ and the perturbation g^1, is therefore a uniform contraction on ξ for small enough ε, and for ξ in a sufficiently small neighbourhood around zero in $B(\mathbb{R})$. Suppose that the contraction constant with respect to ξ for this operator is $\vartheta \in (0, 1)$. By the contraction mapping principle of Banach–Caccipoli (see, for example, [8]), this implies that (2.7) has a unique solution $\bar{\xi}(\tau, \varepsilon)(t)$ for small enough ε. Recall that g^1 is of order $\mathrm{O}(\varepsilon)$ by hypothesis 2. Hence, $T(\xi, \varepsilon)$ satisfies

$$|T(\xi, \varepsilon) - T(\xi, 0)| \leqslant C|\varepsilon|$$

for some positive constant C. Consider the solution $\xi(\varepsilon)$ of (2.7). Since $\xi(0) = 0$, we have

$$|\xi(\varepsilon)| = |T(\xi(\varepsilon), \varepsilon)| \leqslant |T(\xi(\varepsilon), \varepsilon) - T(\xi(0), \varepsilon)| + |T(0, \varepsilon) - T(0, 0)| \leqslant \vartheta|\xi(\varepsilon)| + C|\varepsilon|$$

for some positive constant C, where the last step is because $T(\xi, \varepsilon)$ is a uniform contraction in ξ and is small in ε. Hence,

$$|\xi(\varepsilon)| \leqslant \frac{C}{1-\vartheta}|\varepsilon|$$

and the solution $\bar{\xi}(\tau, \varepsilon)(t)$ of (2.7) is small in ε. Now, the existence of a heteroclinic point is equivalent to the existence of solutions to equations (2.6) and (2.7) as has been described; theorem 3.3 of [8] discusses this fact in greater detail. Therefore, a heteroclinic point exists in the neighbourhood of $W^U_{A_0} = W^S_{B_0}$ for $\varepsilon \in \mathcal{I}$ if and only if there is a solution to (2.6) or equivalently, if there exists τ and $\varepsilon \in \mathcal{I}$ such that $d(\tau, \varepsilon) = 0$. The proof of transversality is analogous to that given in [8]. □

An expansion of the distance function $d(\tau, \varepsilon)$ is given in the following corollary.

Corollary 1. *Suppose all the assumptions of theorem 1 are met, and write*

$$g^1(u, t, \varepsilon) = \varepsilon\tilde{g}^1(u, t, \varepsilon).$$

The distance function can then be written in the form

$$\begin{aligned} d(\tau, \varepsilon) &= \varepsilon M(\tau, \varepsilon) + \mathrm{O}(\varepsilon^2) \\ M(\tau, \varepsilon) &= \int_{-\infty}^{\infty} \langle\varphi(t), \tilde{g}^1(\bar{u}(t), t + \tau, \varepsilon)\rangle\, \mathrm{d}t. \end{aligned} \tag{2.9}$$

Proof. Since we have sufficient smoothness in g^0 and g^1,

$$\begin{aligned} g^0(\bar{u}(t) + \xi) - g^0(\bar{u}(t)) - \nabla g^0(\bar{u}(t))\xi &= \mathrm{O}(|\xi|^2) \\ g^1(\bar{u}(t) + \xi, t + \tau, \varepsilon) &= g^1(\bar{u}(t), t + \tau, \varepsilon) + \mathrm{O}(\varepsilon|\xi|) \end{aligned}$$

using hypothesis 2. However, theorem 1 asserts that $\xi(\varepsilon) = \mathrm{O}(\varepsilon)$ holds. Thus, $d(\tau, \varepsilon)$ has the required form. □

We call $d(\tau, \varepsilon)$ the distance function, while the leading-order term $M(\tau, \varepsilon)$ is referred to as the Melnikov function. It is useful to note that the function $d(\tau, \varepsilon)$ measures a *signed* distance between perturbed stable and unstable manifolds. Indeed, let $u^U_A(\tau, \varepsilon)(t)$ and $u^S_B(\tau, \varepsilon)(t)$ be trajectories in the unstable manifold of $A_\varepsilon(t)$ and the stable manifold of $B_\varepsilon(t)$ for equation (2.3), respectively, with the property that their scalar product with $g^0(\bar{u}(0))$ vanishes at $t = 0$. Then,

$$d(\tau, \varepsilon) = \langle\varphi(0), u^U_A(\tau, \varepsilon)(0) - u^S_B(\tau, \varepsilon)(0)\rangle. \tag{2.10}$$

In particular, the sign of $d(\tau, \varepsilon)$ indicates the direction in which the heteroclinic connection is broken.

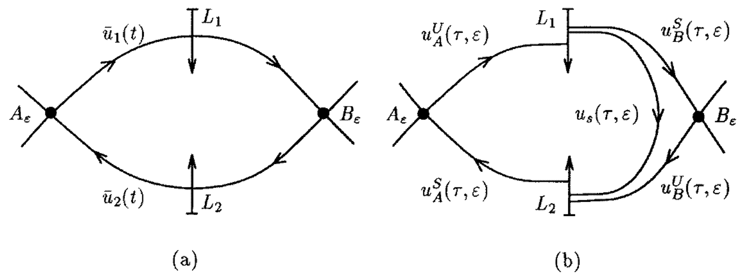

Figure 3. (*a*) Unperturbed and (*b*) perturbed cat's eye.

3. Distance between invariant manifolds near perturbed heteroclinic loops

As indicated in figure 2, a key component to a meandering jet are the flanking cat's eyes, that is, two heteroclinic orbits forming a loop as depicted in figure 3(*a*). Chaotic transport may occur if the loop is broken. In this section, estimates for the distances between stable and unstable manifolds in a perturbed cat's eye are presented. Since equations (1.7) for Lagrangian trajectories are Hamiltonian, we define

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and consider

$$\dot{u} = J\nabla(h^0(u) + h^1(u, t + \tau, \varepsilon)) \tag{3.1}$$

for $u \in \Omega$. The functions h^0 and h^1 satisfy the following hypothesis.

Hypothesis 3. *The nonlinearities h^0 and h^1 are C^{r+1} for some $r \geqslant 2$. Furthermore, $\nabla h^0(u)$ vanishes at most at isolated points in Ω, and $g^1 := J\nabla h^1(u, t, \varepsilon)$ satisfies hypothesis 2.*

Next, we assume that (3.1) has a cat's-eye structure.

Hypothesis 4. *For $\varepsilon = 0$, there exist hyperbolic equilibria A_0 and B_0 of (3.1) such that branches of their one-dimensional stable and unstable manifolds $W^U_{A_0}$ and $W^S_{B_0}$ as well as $W^U_{B_0}$ and $W^S_{A_0}$, respectively, coincide forming a cat's eye for the unperturbed flow of (3.1). Denote the heteroclinic trajectories by $\bar{u}_1(t)$ and $\bar{u}_2(t)$, respectively, see figure 3(a).*

Under these assumptions, we may therefore apply the theory developed in the previous section for each of the two heteroclinic orbits $\bar{u}_j(t)$ with $j = 1, 2$. By lemma 1, bounded solutions $\varphi_j(t)$ of the adjoint equation (2.2) along $\bar{u}_j(t)$ are given by $\nabla h^0(\bar{u}_j(t))$. It then follows that $\nabla h^0(\bar{u}_j(0))$ is not zero since otherwise $\nabla h^0(\bar{u}_j(t))$ would vanish for all t contradicting hypothesis 3. Therefore, $\varphi_j(t)$ is not the trivial zero solution.

We are interested in estimates for the distances between stable and unstable manifolds of $A_\varepsilon(t)$ for small non-zero ε, see figure 5(*b*). Denote the eigenvalues of $\nabla^2 h^0(B_0)$ by $\pm\lambda$ with $\lambda > 0$. Define sections L_1 and L_2 by

$$L_j = \{u \in \Omega | \bar{u}_j(0) - u \in \operatorname{span} \nabla h^0(\bar{u}_j(0)), |\bar{u}_j(0) - u| < \delta\}$$

for some small $\delta > 0$ and $j = 1, 2$. Let $u^S_A(\tau, \varepsilon)(t)$ and $u^U_A(\tau, \varepsilon)(t)$ be trajectories in the stable and unstable manifold of $A_\varepsilon(t)$ for equation (3.1) such that $u^S_A(\tau, \varepsilon)(0) \in L_2$ and $u^U_A(\tau, \varepsilon)(0) \in L_1$, see figure 3. Similarly, $u^S_B(\tau, \varepsilon)(t)$ and $u^U_B(\tau, \varepsilon)(t)$ denote trajectories contained in the perturbed stable and unstable manifolds of $B_\varepsilon(t)$ satisfying $u^S_B(\tau, \varepsilon)(0) \in L_1$ and $u^U_B(\tau, \varepsilon)(0) \in L_2$, respectively. Let $\gamma > 0$ be arbitrary but fixed.

The next lemma gives a determination of any solution starting in L_1 and ending in L_2 in terms of the time needed to pass the solution $B_\varepsilon(t)$.

Lemma 3. *Suppose hypotheses 3 and 4 are met, then there exist $\varepsilon_0 > 0$ and s_0 such that for any τ, ε and s with $|\varepsilon| < \varepsilon_0$ and $s > s_0$ the following holds: there is a unique solution $u_s(\tau,\varepsilon)(t)$ of equation (3.1) defined for $t \in [0, 2s]$ such that*

$$u_s(\tau,\varepsilon)(0) \in L_1 \qquad \textit{and} \qquad u_s(\tau,\varepsilon)(2s) \in L_2. \tag{3.2}$$

Moreover,

$$\begin{aligned}\langle \nabla h^0(\bar u_1(0)), u_A^U(\tau,\varepsilon)(0) - u_s(\tau,\varepsilon)(0)\rangle \\ = d_1(\tau,\varepsilon) - \langle \nabla h^0(\bar u_1(s)), \bar u_2(-s) - B_0\rangle + R_1(\tau,\varepsilon,s)\end{aligned} \tag{3.3}$$

$$\begin{aligned}\langle \nabla h^0(\bar u_2(0)), u_s(\tau,\varepsilon)(2s) - u_A^S(\tau+2s,\varepsilon)(0)\rangle \\ = d_2(\tau+2s,\varepsilon) + \langle \nabla h^0(\bar u_2(-s)), \bar u_1(s) - B_0\rangle + R_2(\tau+2s,\varepsilon,s)\end{aligned} \tag{3.4}$$

where the remainder terms satisfy the estimate

$$|R_j(\tau,\varepsilon,s)| \leqslant C_\gamma(|\varepsilon| + \mathrm{e}^{-2\lambda s})\mathrm{e}^{-\lambda(1-\gamma)s} \tag{3.5}$$

for any $\gamma > 0$. Moreover, $d_j(\tau,\varepsilon)$, $j = 1, 2$ are the distance functions for the two intersections of stable and unstable manifolds defined in the previous section computed with respect to $\varphi_j(t) = \nabla h^0(\bar u_j(t))$.

Proof. Existence and uniqueness follow as in [17]. In [17] only autonomous, smooth perturbations are considered. It is straightforward to adapt the proof given there to the situation studied here. We therefore refer to [17] for details. □

Note that we do not claim any smoothness properties for $u_s(\tau,\varepsilon)$ nor the remainder terms R_j. By [28, lemma 1.1], we have

$$\begin{aligned}\langle \nabla h^0(\bar u_1(s)), \bar u_2(-s) - B_0\rangle &= K_1\mathrm{e}^{-2\lambda s} + R_3(s)\\ \langle \nabla h^0(\bar u_2(-s)), \bar u_1(s) - B_0\rangle &= K_2\mathrm{e}^{-2\lambda s} + R_4(s)\end{aligned} \tag{3.6}$$

for some positive constants K_1 and K_2, and, for any small $\gamma > 0$,

$$|R_3(s)| + |R_4(s)| \leqslant C_\gamma \mathrm{e}^{-3\lambda(1-\gamma)s}$$

holds as $s \to \infty$. We will need the relation $K_1 = K_2$ proved in the next lemma.

Lemma 4. *Under the assumptions of lemma 3, $K_1 = K_2$.*

Proof. The constants K_1 and K_2 are determined by the unperturbed flow for $\varepsilon = 0$. Let $u_s(\tau, 0)(t) =: u_s(t)$ be independent of τ. We then have

$$\begin{aligned}h^0(\bar u_1(0)) - h^0(u_s(0)) &= \nabla h^0(\bar u_1(0)) \cdot (\bar u_1(0) - u_s(0)) + \mathrm{O}(|\bar u_1(0) - u_s(0)|^2)\\ &= \langle \nabla h^0(\bar u_1(0)), \bar u_1(0) - u_s(0)\rangle + \mathrm{O}(|\bar u_1(0) - u_s(0)|^2)\\ &= -K_1\mathrm{e}^{-2\lambda s} + \mathrm{O}(\mathrm{e}^{-3\lambda(1-\gamma)s})\end{aligned}$$

and similarly obtain

$$\begin{aligned}\nabla h^0(\bar u_2(0)) \cdot (\bar u_2(0) - u_s(2s)) &+ \mathrm{O}(|\bar u_2(0) - u_s(2s)|^2)\\ &= -\langle \nabla h^0(\bar u_2(0)), u_s(2s) - \bar u_2(0)\rangle + \mathrm{O}(|\bar u_2(0) - u_s(2s)|^2)\\ &= -K_2\mathrm{e}^{-2\lambda s} + \mathrm{O}(\mathrm{e}^{-3\lambda(1-\gamma)s}).\end{aligned}$$

Since h^0 is a conserved quantity for equation (3.1) with $\varepsilon = 0$, we conclude

$$h^0(\bar{u}_1(0)) - h^0(u_s(0)) = h^0(\bar{u}_2(0)) - h^0(u_s(2s)).$$

Indeed, $h^0(u_s(0)) = h^0(u_s(2s))$ and $h^0(\bar{u}_1(0)) = h^0(\bar{u}_2(0))$ hold. Therefore, $K_1 = K_2$ by choosing γ sufficiently small. □

The particular form $J\nabla h^1(u, t, \varepsilon)$ of the perturbation guarantees that the flow is area-preserving. Next, we assume that the splitting of stable and unstable manifolds is, to first order, independent of the time slice. This assumption is very restrictive, since we expect transverse intersections of the invariant manifolds for almost any perturbation. However, it is often enforced in the application to oceanography we are interested in, since then only those perturbations are allowed which are solutions of the vorticity equation, see section 8.

Hypothesis 5. *Suppose that $d_j(\tau, \varepsilon) = \varepsilon M_j(\varepsilon) + \mathrm{O}(\varepsilon^2)$, where $M_j(\varepsilon)$ is independent of τ.*

It then follows that the splitting distances coincide to first order.

Lemma 5. *Suppose that hypotheses 3–5 are satisfied, then $|M_1(\varepsilon) + M_2(\varepsilon)| \leqslant C|\varepsilon|$.*

Proof. Note that $\mathrm{div}(J\nabla(h^0(u) + h^1(u, t, \varepsilon))) = 0$ vanishes identically, and therefore the flow of (3.1) is area-preserving. Denote the area enclosed by the unperturbed cat's eye by V_0. Then, the area $V_\varepsilon(\tau)$ enclosed by the perturbed cat's eye in the time slice $\{t = \tau\}$ can be estimated by

$$|V_\varepsilon(\tau) - V_0| \leqslant C\varepsilon \tag{3.7}$$

uniformly in ε and τ. Here, the perturbed cat's eye refers to the pieces $u_A^U(\tau+s, \varepsilon)(-s)$ and $u_B^U(\tau + s, \varepsilon)(-s)$ as well as $u_A^S(\tau - s, \varepsilon)(s)$ and $u_B^S(\tau - s, \varepsilon)(s)$ of the invariant manifolds in the time slice $\{t = \tau\}$ together with the pieces of the sections L_j connecting them, see figure 3(*b*). Here, $s \in \mathbb{R}^+$.

Note that the gradients $\nabla h^0(\bar{u}_1(0))$ and $\nabla h^0(\bar{u}_2(0))$ point either both inside the interior of the cat's eye, or else both outside, since the equilibria are hyperbolic, that is $\nabla^2 h^0(A_0)$ is invertible. Set $\kappa = 1$ if both point inside, and set $\kappa = -1$ if both point outside. We shall quantify the effects of volume leaving or entering the perturbed cat's eye. The distances of stable and unstable manifolds in the time slice $\{t = \tau\}$ measured in the direction of $\nabla h^0(\bar{u}_j(0))$ are given by

$$D_1 := u_A^U(\tau, \varepsilon)(0) - u_B^S(\tau, \varepsilon)(0) = \varepsilon\kappa M_1(\varepsilon)|\nabla h^0(\bar{u}_1(0))|^{-1} + \mathrm{O}(\varepsilon^2)$$
$$D_2 := u_B^U(\tau, \varepsilon)(0) - u_A^S(\tau, \varepsilon)(0) = \varepsilon\kappa M_2(\varepsilon)|\nabla h^0(\bar{u}_2(0))|^{-1} + \mathrm{O}(\varepsilon^2).$$

by lemma 3 and hypothesis 5. In particular, $\kappa M_j < 0$ and $\kappa M_j > 0$ correspond to volume leaving and entering the perturbed cat's eye through the section L_j, respectively, for $j = 1, 2$. For small time intervals of length T, the amount of volume flowing through the section L_j is given by

$$\partial_j V_\varepsilon(\tau) := D_j T(|\nabla h^0(\bar{u}_j(0))| + \mathrm{O}(\varepsilon)).$$

Substituting the expression for D_j, we obtain

$$\partial_j V_\varepsilon(\tau) = \varepsilon\kappa T(M_j(\varepsilon) + \mathrm{O}(\varepsilon)) \tag{3.8}$$

where the sign of $\partial_j V_\varepsilon(\tau)$ indicates whether volume is entering or leaving the perturbed cat's eye. However, since $M_j(\varepsilon)$ is independent of τ, this amount of volume keeps adding up by considering several disjoint time intervals of length T, and it cannot be compensated by a change of the area due to the remainder term $\mathrm{O}(\varepsilon)$ in (3.7). Therefore, area conservation is only possible if $M_1(\varepsilon) = -M_2(\varepsilon) + \mathrm{O}(\varepsilon)$. □

In section 2, the intersections between the unstable manifold of A_ε and the stable manifold of B_ε were analysed. Here, we are interested in intersections of the unstable and stable manifolds of A_ε, that is, orbits homoclinic to A_ε, see figure 5(*b*) for the geometry. Since the unstable manifold of A_ε would then pass near B_ε, these homoclinic solutions are reminiscent of multiple or secondary homoclinic orbits in autonomous systems, see, for instance, [12, 15, 28, 29]. After the separatrices $\bar{u}_j(t)$ are broken for $\varepsilon \neq 0$, the unstable manifold of A_ε may pass near B_ε and intersect with the stable manifold of A_ε. We therefore define

$$d_{\mathrm{hom}}(\tau,\varepsilon) := \langle \nabla h^0(\bar{u}_2(0)), u_A^U(\tau,\varepsilon)(2s_*) - u_A^S(\tau+2s_*,\varepsilon)(0)\rangle$$

where s_* is chosen such that $u_A^U(\tau,\varepsilon)(t)$ intersects L_2 for the first time at $t = 2s_*$. The quantity $d_{\mathrm{hom}}(\tau,\varepsilon)$ measures the distance between stable and unstable manifolds of the solution A_ε at the section L_2 in the time slice $\{t = \tau + 2s_*\}$. Of course, the unstable manifold $u_A^U(\tau,\varepsilon)(t)$ of A_ε may not intersect L_2 at all; however, whenever it does, the quantity $d_{\mathrm{hom}}(\tau,\varepsilon)$ is well defined. We then have the following result which shows that the intersections associated with an orbit homoclinic to A_ε occur only at higher order.

Theorem 2. *Assume that hypothesis 3–5 are met. Fix some* $\nu \in (0, \frac{1}{2})$, *then, whenever* $d_{\mathrm{hom}}(\tau,\varepsilon)$ *is defined,* $|d_{\mathrm{hom}}(\tau,\varepsilon)| \leqslant C_\nu |\varepsilon|^{1+\nu}$.

Proof. By lemma 5, $M_1(\varepsilon) = M_2(\varepsilon) + \mathrm{O}(\varepsilon)$. We set $M(\varepsilon) := M_1(\varepsilon)$. Consider equation (3.3)

$$\begin{aligned}&\langle \nabla h^0(\bar{u}_1(0)), u_A^U(\tau,\varepsilon)(0) - u_s(\tau,\varepsilon)(0)\rangle \\ &\quad = d_1(\tau,\varepsilon) - \langle \nabla h^0(\bar{u}_1(s)), \bar{u}_2(-s) - B_0\rangle + R_1(\tau,\varepsilon,s) = 0. \end{aligned} \tag{3.9}$$

Substituting (3.3), (3.6) and using lemma 5, we obtain the equation

$$\varepsilon M(\varepsilon) - K_1 \mathrm{e}^{-2\lambda s} + R_5(\tau,\varepsilon,s) = 0 \tag{3.10}$$

for some remainder term satisfying

$$|R_5(\tau,\varepsilon,s)| \leqslant C_\gamma(|\varepsilon|^2 + (|\varepsilon| + \mathrm{e}^{-2\lambda s})\mathrm{e}^{-\lambda(1-\gamma)s}).$$

On account of the uniqueness statement in lemma 3, the unstable manifold of A_ε will intersect L_2 if and only if (3.10) has a solution. Then the solutions $u_A^U(\tau,\varepsilon)(t)$ and $u_s(\tau,\varepsilon)(t)$ coincide and we can therefore use equation (3.4) to estimate $d_{\mathrm{hom}}(\tau,\varepsilon)$. Assuming that a solution $(\tau,\varepsilon,s) = (\tau_*,\varepsilon_*,s_*)$ of (3.10) has been found, we obtain the estimate

$$\mathrm{e}^{-2\lambda s_*} \leqslant C|\varepsilon_*| \tag{3.11}$$

by inspecting (3.10). It remains to estimate

$$\begin{aligned} d_{\mathrm{hom}}(\tau_*,\varepsilon_*) &= \langle \nabla h^0(\bar{u}_2(0)), u_{s_*}(\tau_*,\varepsilon_*)(2s_*) - u_A^S(\tau_*+2s_*,\varepsilon_*)(0)\rangle \\ &= -\varepsilon_* M(\varepsilon_*) + K_2 \mathrm{e}^{-2\lambda s_*} + R_6(\tau_*,\varepsilon_*,s_*) \end{aligned} \tag{3.12}$$

where we substituted (3.4), (3.6) and used lemma 5. Here,

$$|R_6(\tau,\varepsilon,s)| \leqslant C_\gamma(|\varepsilon|^2 + (|\varepsilon| + \mathrm{e}^{-2\lambda s})\mathrm{e}^{-\lambda(1-\gamma)s}).$$

Adding (3.10) evaluated at $(\tau_*,\varepsilon_*,s_*)$ and (3.12) yields

$$d_{\mathrm{hom}}(\tau_*,\varepsilon_*) = R_5(\tau_*,\varepsilon_*,s_*) + R_6(\tau_*,\varepsilon_*,s_*).$$

Indeed, $K_1 = K_2$ holds by lemma 4. Thus, using estimate (3.11), we obtain

$$|d_{\mathrm{hom}}(\tau_*, \varepsilon_*)| \leqslant C_\gamma(|\varepsilon_*|^2 + (|\varepsilon_*| + \mathrm{e}^{-2\lambda s_*})\mathrm{e}^{-\lambda(1-\gamma)s_*})$$
$$\leqslant C_\gamma(|\varepsilon_*|^2 + |\varepsilon_*|^{1+\frac{1}{2}(1-\gamma)}) \leqslant C_\nu|\varepsilon_*|^{1+\nu}$$

for any $\nu < \frac{1}{2}$. □

4. Viscous dynamics and vanishing viscosity

Throughout, we use the variables x and y as being those on the barotropic β-plane that define the eastward and northward directions respectively and let $z = (x, y)$. Suppose now that the dynamics of the barotropic jet are governed not by the exact conservation of potential vorticity but by a potential vorticity dissipating, forced flow. The dynamics will thus be assumed to satisfy

$$\frac{\mathrm{D}q}{\mathrm{D}t} \equiv \frac{\partial q}{\partial t} - \frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} = \varepsilon[\Delta q + f(x, y, t)] \tag{4.1}$$

where $q(x, y, t) = \Delta\psi(x, y, t) + \beta y$ is the barotropic potential vorticity, $f(x, y, t)$ is a uniformly bounded forcing function, and ψ is the streamfunction associated with the flow. The parameter ε lies in $(0, \varepsilon_0]$, where ε_0 is assumed to be as small as needed, and represents the viscosity. The above dissipative dynamics can result from directly including the Newtonian viscosity in the primitive equations or, more realistically in the oceanographic context, by the dissipation caused by eddy diffusivity (the averaged effects of small scale turbulence). In either case, ε can be considered as a small parameter for oceanic flows. The function $f(x, y, t)$ can be thought of as modelling wind forcing.

It is relevant to find out whether the flow of this dissipative equation is close to that of the exactly potential vorticity conserving equation

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0. \tag{4.2}$$

This issue has been addressed in [16] for $\beta = 0$. In [1, 2], this problem was investigated for $\beta \neq 0$. Recall that q is related to the associated streamfunction by $q = \Delta\psi + \beta y$. Suppose the streamfunction $\psi^0(x, y, t)$ satisfies (4.2), while $\psi(x, y, t; \varepsilon)$ obeys (4.1). Let $(x, y) \in \Omega$, a two-dimensional smooth surface *with no boundary*. The traditional β-plane, which is $\mathbb{R}^2$, obeys this constraint as does a torus and an infinite cylinder which can be used via imposition of periodic boundary conditions. Suppose the initial conditions $\nabla\psi(x, y, 0; \varepsilon)$ and $\nabla\psi^0(x, y, 0)$ are $\mathrm{O}(\varepsilon)$ close in the norm $C^3(\Omega)$ and in the Sobolev norms $H^3(\Omega)$ and $H^4(\Omega)$. Let $T > 0$ be large but fixed, and suppose it is known that the inviscid streamfunction is smooth enough so that

$$\sum_{5\leqslant|k|\leqslant 7} \|D^k\psi^0(t)\|_{L^2(\Omega)} \qquad \text{and} \qquad \sum_{4\leqslant|k|\leqslant 7} \|D^k\psi^0(t)\|_{L^4(\Omega)}$$

are bounded independently of $t \in [0, T]$. The generalized derivative symbol D^k used here is assumed to act only on the spatial (x, y) variables. Then, it can be shown that, see [1, 2], there exists a constant $C(T)$ such that

$$\sup_{t\in[0,T]} \|\nabla\psi(t; \varepsilon) - \nabla\psi^0(t)\|_{C^3(\Omega)} \leqslant \varepsilon C(T). \tag{4.3}$$

If the additional smoothness assumption on the inviscid streamfunction is removed, the above can be derived in the norm of $C^0(\Omega)$. Thus, the velocity field of the viscous dynamics is $\mathrm{O}(\varepsilon)$ close to that of its inviscid counterpart. The derivation of (4.3) involves extensive use of *a priori* estimates and the Sobolev embedding theorem [1, 2]. From (4.3), it is possible to write

$$\nabla\psi(x, y, t; \varepsilon) = \nabla\psi^0(x, y, t) + \varepsilon\nabla\psi^1(x, y, t; \varepsilon) \tag{4.4}$$

for any $t \in [0, T]$ and $(x, y) \in \Omega$ such that $\sum_{0\leqslant|k|\leqslant 4} D^k\psi^1(x, y, t; \varepsilon)$ is bounded independently of $\varepsilon \in [0, \varepsilon_0]$ for finite times.

For smooth, real-valued functions $f, g : \Omega \to \mathbb{R}$, the *Poisson bracket* between f and g is defined by

$$\{f, g\} := \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

Using this definition, substitution of (4.4) into the dynamics (4.1) leads to the equation

$$\frac{\mathrm{D}^0 q^1}{\mathrm{D}t} + \{\psi^1, q^0\} = \Delta q^0 + f + \varepsilon[\Delta q^1 - \{\psi^1, q^1\}] \tag{4.5}$$

where the unperturbed material derivative

$$\frac{\mathrm{D}^0}{\mathrm{D}t} = \frac{\partial}{\partial t} - \frac{\partial\psi^0}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\psi^0}{\partial x}\frac{\partial}{\partial y}$$

has been used, along with the notation

$$q^0 = \Delta\psi^0 + \beta y \qquad \text{and} \qquad q^1 = \Delta\psi^1.$$

Note that (4.4) ensures that all terms in (4.5) remain bounded for finite times. The Lagrangian trajectories generated by the inviscid streamfunction ψ^0 satisfy the differential equation

$$\dot{z} = J\nabla\psi^0(z, t) \tag{4.6}$$

with $z = (x, y)$ and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This possesses $q^0(x, y, t)$ as an integral of motion. In fact, (4.6) is formally integrable [7]. We additionally assume that (4.6) possesses the gross kinematics of an oceanic jet in that heteroclinic trajectories exist between two saddle structures, see figure 2. Mathematically speaking, we impose hypothesis 1 on the flow (4.6). Such a heteroclinic represents the boundary of a cat's eye, and its destruction would permit fluid to travel across these apparent separatrices.

To apply the Melnikov theory developed in section 2, we need to consider a perturbation to the vector field in (4.6). This is readily accomplished by using instead the full viscous streamfunction

$$\psi(x, y, t; \varepsilon) = \psi^0(x, y, t) + \varepsilon\psi^1(x, y, t; \varepsilon) \tag{4.7}$$

and thus the Lagrangian trajectories of the perturbed flow will obey

$$\dot{z} = J\nabla\psi(z, t; \varepsilon). \tag{4.8}$$

It is known by (4.3) that, for finite times at least, the vector field of (4.8) is $\mathrm{O}(\varepsilon)$ close to that of the integrable system (4.6), i.e. the first-order term $\psi^1(x, y, t; \varepsilon)$ is bounded. Assuming that this closeness can be extended to all $t \in \mathbb{R}$, the difference in the velocity fields of (4.8) and (4.6) would satisfy hypothesis 2. We are then in a position to use the distance function $d(\tau, \varepsilon)$, developed in section 2, to investigate intersections of the perturbed manifolds, that is persistence of heteroclinic points.

5. Assumptions and simplifications

We now further restrict our attention to a particular class of flows that we call *shifted autonomous flows*. These are frequently used in modelling potential vorticity-conserving flows [4, 11, 19, 20, 24–26]; in fact, to our knowledge, there are no known analytical, or closed-form solutions which are not shifted autonomous while conserving barotropic potential vorticity. The time dependence of these flows can be removed by transforming coordinates to a moving frame. Alternatively, a shifted autonomous flow is a travelling-wave solution of speed c.

Assumption 1. *Equation (4.6) is* shifted autonomous*; that is, there exists c and a function $\Psi^0(\xi,\eta)$ such that $\psi^0(x,y,t) = \Psi^0(x-ct,y) = \Psi^0(\xi,\eta)$ where $\xi = x-ct$ and $\eta = y$. The change of variables $(x,y,t) \to (\xi,\eta,t)$ is called the* shift*.*

It will be shown below that it indeed suffices to consider shifted autonomous flows which travel in the x-direction. We will consistently use (ξ,η) as the shifted variables in what follows. Moreover, the relevant capital letter will be used to denote a variable in the shifted coordinates, for example

$$\psi^0(x,y,t) = \Psi^0(\xi,\eta)$$

since there is no direct t dependence when ψ^0 is shifted to the (ξ,η) coordinates by assumption 1. Furthermore, noting that the spatial derivatives are invariant under a shift, we will use the operators ∇, Δ, etc as operating on either the (x,y) variables or the (ξ,η) variables; on which shall be clear from the context. If assumption 1 is met, equation (4.6) transforms into

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = J\nabla(\Psi^0(\xi,\eta) + c\eta) \tag{5.1}$$

with a Hamiltonian given by $\Psi^0(\xi,\eta) + c\eta$.

Another hypothesis is now imposed on the unperturbed flow, namely, that equation (4.6) in shifted coordinates possesses a structure as depicted in figure 2.

Assumption 2. *Equation (5.1) has a homoclinic trajectory $(\bar{\xi},\bar{\eta})$ connecting the hyperbolic equilibrium (ξ_A,η_A) to itself.*

We should comment on assumption 2. The structure shown in figure 2 is formed by two heteroclinic orbits connecting two different equilibria with each other, and not by a homoclinic solution. However, the structure is periodic in the easterly direction. Therefore, we can view the equation on the infinite cylinder $\Omega = S^1 \times \mathbb{R}$ or the torus $\Omega = S^1 \times S^1$ rather than on the usual β-plane $\Omega = \mathbb{R}^2$. With this choice of the domain Ω, the two different equilibria appearing in figure 2 are then identified. Many models arising in the literature are periodic in one of the spatial coordinates and therefore allow for such a reduction. Notice, however, that then the forcing term $F(\xi,\eta,t)$ has also to be periodic in the spatial variables. Indeed, the partial differential equation (4.1) is then considered on a domain where one or both spatial variables are periodic and the forcing term must be defined on the same domain.

Assumption 2 has been expressed in the shifted coordinates. For $\tau \in \mathbb{R}$, let

$$\bar{z}(t+\tau;\tau) = (\bar{x}(t+\tau;\tau), \bar{y}(t+\tau;\tau)) := (\bar{\xi}(t) + c(t+\tau), \bar{\eta}(t)) \tag{5.2}$$

be the corresponding solution of the original equation (4.6)

$$\dot{z} = J\nabla\psi^0(z, t+\tau)$$

with $z(0) = (\bar{\xi}(0) + c\tau, \bar{\eta}(0))$.

We now require that, in addition to assumptions 1 and 2 on the unperturbed flow, the perturbation also satisfies a constraint.

Assumption 3. *The function* $\Psi^1 \in C^4(\Omega)$ *is bounded in* $C^4(\Omega)$ *uniformly in* $t \in \mathbb{R}$ *and* $\varepsilon \in (0, \varepsilon_0]$. *Here,* Ψ^1 *denotes the function* ψ^1 *satisfying (4.5) in shifted coordinates.*

Recall that the claims implicit in assumption 3 have only been proven for *finite* times. The boundedness assumption 3 will ensure that (4.3)–(4.5) are valid for all $t \in \mathbb{R}$. Note that assumption 3 is met whenever the perturbed streamfunction is periodic in time. In section 7 we shall comment on whether assumption 3 is satisfied by equation (4.1). Equation (4.8) in shifted coordinates reads

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = J\nabla(\Psi^0(\xi, \eta) + c\eta + \varepsilon\Psi^1(\xi, \eta, t; \varepsilon)). \tag{5.3}$$

It turns out that, if assumption 2 is met, the travelling waves we are considering must, in fact, travel in an easterly direction.

Lemma 6. *Assume that* $\psi^0(x, y, t)$ *satisfies (4.6) and that* $\psi^0(x, y, t) = \Psi^0(\xi, \eta)$ *where* $\xi = x - c_x t$ *and* $\eta = y - c_y t$. *If assumption 2 is met with (5.1) replaced by*

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = J\nabla(\Psi^0(\xi, \eta) - c_y\xi + c_x\eta) \tag{5.4}$$

then $\beta c_y = 0$.

Proof. We observe that the potential vorticity $q^0(x, y, t)$ in shifted coordinates is given by

$$Q^0(\xi, \eta, t) = \Delta\Psi^0(\xi, \eta) + \beta(\eta + c_y t). \tag{5.5}$$

However,

$$\frac{\mathrm{d}}{\mathrm{d}t} Q^0(\xi(t), \eta(t), t) = \frac{\mathrm{d}}{\mathrm{d}t} q^0(x(t), y(t), t) = 0 \tag{5.6}$$

along any solution $(\xi(t), \eta(t))$ of (5.4), since q^0 is conserved along trajectories. Picking the equilibrium (ξ_A, η_A) which exists by assumption 2, we then see that the left-hand side of (5.6) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} Q^0(\xi_A, \eta_A, t) = \beta c_y$$

by evaluating (5.5). Therefore, by (5.6), we obtain $\beta c_y = 0$. □

If $\beta = 0$, we may always rotate coordinates to obtain $c_y = 0$, since equation (4.2) is then invariant under rotations in the (x, y)-plane.

6. Distance function computation

The key computation is contained in this section. An exact expression for the Melnikov function, the first-order term in the distance function, will be derived. The surprising fact is that the expression is given *explicitly* in terms of the $\varepsilon = 0$ velocity field, in fact the inviscid potential vorticity, and the forcing. As indicated in the introduction, a Melnikov function calculation usually involves the perturbed velocity field, see equation (1.8), and this is unknown here. However, with the form of perturbed PDE we are considering, i.e. perturbation by viscosity and forcing, we do not need to know the perturbed flow field exactly.

Theorem 3. *Suppose that the unperturbed flow (4.6) satisfies the shifted autonomous assumption 1. Suppose that equation (5.1), that is (4.6) in shifted coordinates, obeys assumption 2. Let the dynamics (4.1) generate the perturbation ψ^1, which in shifted coordinates is assumed to obey assumption 3. Then the distance function for equation (5.3) computed with respect to $\nabla Q^0(\bar{\xi}(0), \bar{\eta}(0))$ has the form*

$$d(\tau, \varepsilon) = \varepsilon M(\tau) + \mathrm{O}(\varepsilon^2) \tag{6.1}$$

with

$$M(\tau) = \int_{-\infty}^{\infty} [\Delta Q^0(\bar{\xi}(t), \bar{\eta}(t)) - \Delta Q^0(\xi_A, \eta_A)]\,\mathrm{d}t + \int_{-\infty}^{\infty} [F(\bar{\xi}(t), \bar{\eta}(t), t+\tau) - F(\xi_A, \eta_A, t+\tau)]\,\mathrm{d}t \tag{6.2}$$

where the upper case notation corresponds to the shifted variables.

Note that the function $M(\tau)$, as defined above, coincides with the Melnikov function $M(\tau, \varepsilon)$ defined in corollary 1 up to order $\mathrm{O}(\varepsilon)$. For this reason, with a slight abuse of notation, we will also refer to $M(\tau)$ as the Melnikov function since it constitutes the first-order term in the expression for the distance function $d(\tau, \varepsilon)$.

Proof. Note that the theory developed in section 2 is applicable to equation (5.3), which is (4.8) in shifted coordinates, as hypothesis 2 is met on account of the boundedness assumption 3 and the results stated in section 4. Furthermore, assumption 3 ensures that the dynamics (4.5) is satisfied for all $t \in \mathbb{R}$. Notice that assumption 2 gives us at the outset that $A_0 = B_0$.

We observe that the potential vorticity $q^0(x, y, t)$ in shifted coordinates

$$Q^0(\xi, \eta) = \Delta\Psi^0(\xi, \eta) + \beta\eta$$

is time independent. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t} Q^0(\xi(t), \eta(t)) = \frac{\mathrm{d}}{\mathrm{d}t} q^0(x(t), y(t), t) = 0$$

along any solution $(\xi(t), \eta(t))$ of (5.1). Therefore, by lemma 1, we have $\varphi(t) = \nabla Q^0(\bar{\xi}(t), \bar{\eta}(t))$.

Hence, on account of corollary 1, the Melnikov function for (5.3), where ψ^1 has been defined in (4.7), is given by

$$\begin{aligned} M(\tau, \varepsilon) &= \int_{-\infty}^{\infty} \nabla Q^0(\bar{\xi}(t), \bar{\eta}(t)) \cdot J\nabla\Psi^1(\bar{\xi}(t), \bar{\eta}(t), t+\tau; \varepsilon)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \nabla q^0(\bar{z}(t+\tau; \tau), t+\tau) \cdot J\nabla\psi^1(\bar{z}(t+\tau; \tau), t+\tau; \varepsilon)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \{\psi^1, q^0\}(\bar{z}(t; \tau), t; \varepsilon)\,\mathrm{d}t \end{aligned} \tag{6.3}$$

by transforming back to the original coordinates, see (5.2), and shifting time.

It is known by (4.3) that $\nabla\psi^1$ is bounded for finite times, and assumption 3 permits the extension to all $t \in \mathbb{R}$. Moreover, ∇q^0 decays exponentially to zero in the integrand as $t \to \pm\infty$, and hence integral (6.3) is absolutely convergent for small enough ε. We know that

$$\nabla q^0(A_0(t), t) = 0$$

for all t where $A_0(t) := (\xi_A + ct, \eta_A)$ denotes the equilibrium in the original coordinates. Therefore, evaluating (4.5) on $(A_0(t), t)$,

$$\frac{\mathrm{D}^0 q^1}{\mathrm{D}t}(A_0(t), t) = \Delta q^0(A_0(t), t) + f(A_0(t), t) + \varepsilon[\Delta q^1(A_0(t), t) - \{\psi^1, q^1\}(A_0(t), t)] \tag{6.4}$$

where the terms multiplying ε are bounded uniformly in t. Since $(A_0(t), t)$ is a trajectory of the unperturbed flow, the material derivative $\frac{\mathrm{D}^0}{\mathrm{D}t}$ is exactly the total derivative evaluated via the chain rule. In other words, note that for any function $h(x, y, t)$ on $\Omega \times \mathbb{R}$, if $(x(t), y(t), t)$ is a trajectory of the unperturbed flow, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}h(x(t), y(t), t) &= \frac{\partial h}{\partial x}\dot{x}(t) + \frac{\partial h}{\partial y}\dot{y}(t) + \frac{\partial h}{\partial t} = \frac{\partial h}{\partial x}\left(-\frac{\partial \psi^0}{\partial y}\right) + \frac{\partial h}{\partial y}\left(\frac{\partial \psi^0}{\partial x}\right) + \frac{\partial h}{\partial t}\\ &= \frac{\mathrm{D}^0}{\mathrm{D}t}h(x(t), y(t), t).\end{aligned} \tag{6.5}$$

Therefore, (6.4) may be written as

$$\frac{\mathrm{d}q^1}{\mathrm{d}t}(A_0(t), t) = \Delta q^0(A_0(t), t) + f(A_0(t), t) + \varepsilon[\Delta q^1(A_0(t), t) - \{\psi^1, q^1\}(A_0(t), t)]. \tag{6.6}$$

Fix $\tau \in \mathbb{R}$, and pick the homoclinic trajectory

$$(\bar{z}(t; \tau), t) = (\bar{x}(t; \tau), \bar{y}(t; \tau), t)$$

of (4.6). We evaluate the dynamical equation (4.5) on this trajectory to obtain

$$\begin{aligned}\{\psi^1, q^0\}(\bar{z}(t; \tau), t) &= \left[\Delta q^0 - \frac{\mathrm{d}q^1}{\mathrm{d}t}\right](\bar{z}(t; \tau), t) + f(\bar{z}(t; \tau), t)\\ &\quad + \varepsilon[\Delta q^1 - \{\psi^1, q^1\}](\bar{z}(t; \tau), t)\end{aligned}$$

where the material derivative following the unperturbed flow has been replaced by $\mathrm{d}/\mathrm{d}t$ by virtue of (6.5). We now add and subtract the quantities $\varepsilon[\Delta q^1 - \{\psi^1, q^1\}](A_0(t), t)$, $\Delta q^0(A_0(t), t)$, and $f(A_0(t), t)$ in appropriate places of the above to obtain

$$\begin{aligned}\{\psi^1, q^0\}(\bar{z}(t; \tau), t) &= [\Delta q^0(\bar{z}(t; \tau), t) - \Delta q^0(A_0(t), t)] + [f(\bar{z}(t; \tau), t) - f(A_0(t), t)]\\ &\quad + \left[(\Delta q^0 + f + \varepsilon[\Delta q^1 - \{\psi^1, q^1\}])(A_0(t), t) - \frac{\mathrm{d}q^1}{\mathrm{d}t}(\bar{z}(t; \tau), t)\right]\\ &\quad + \varepsilon[(\Delta q^1 - \{\psi^1, q^1\})(\bar{z}(t; \tau), t) - (\Delta q^1 - \{\psi^1, q^1\})(A_0(t), t)].\end{aligned} \tag{6.7}$$

The operator $\int_{-\infty}^{0} \mathrm{d}t$ is now applied to the above. The left-hand side yields

$$\int_{-\infty}^{0} \{\psi^1, q^0\}(\bar{z}(t; \tau), t)\, \mathrm{d}t$$

which we recognize as part of the integral defining $M(\tau, \varepsilon)$, see (6.3). We look at each of the three terms in square brackets on the right-hand side separately. We will keep the first, while the second becomes

$$\begin{aligned}&\int_{-\infty}^{0} \left[\Delta q^0(A_0(t), t) + f(A_0(t), t) + \varepsilon[\Delta q^1 - \{\psi^1, q^1\}](A_0(t), t) - \frac{\mathrm{d}q^1}{\mathrm{d}t}(\bar{z}(t; \tau), t)\right] \mathrm{d}t\\ &\qquad = \int_{-\infty}^{0} \frac{\mathrm{d}}{\mathrm{d}t}[q^1(A_0(t), t) - q^1(\bar{z}(t; \tau), t)]\, \mathrm{d}t\\ &\qquad = [q^1(A_0(t), t) - q^1(\bar{z}(t; \tau), t)]_{-\infty}^{0} = Q^1(\xi_A, \eta_A; \varepsilon) - q^1(\bar{z}(0; \tau), 0; \varepsilon).\end{aligned}$$

The first step of the above is by (6.6), while the last is because at the left endpoint, $\bar{z}(t;\tau)$ decays exponentially to $A_0(t)$. This is further facilitated by the knowledge of continuity of q^1 in its spatial variables provided by (4.3). Note that we have transformed the first remaining term involving q^1 back into shifted coordinates. The third term of (6.7) remains $\mathrm{O}(\varepsilon)$ upon integration, since the ε can be extracted from the integral, and the remaining integrand consists of terms known to be uniformly bounded independent of ε. Moreover, the integral is finite, since $\bar{z}(t;\tau)$ decays exponentially to $A_0(t)$ as $t \to -\infty$. Applying the same arguments for $t \geqslant 0$ using the integral operator $\int_0^\infty \mathrm{d}t$ and adding the resulting terms yields

$$M(\tau,\varepsilon) = \int_{-\infty}^{\infty} [\Delta q^0(\bar{z}(t;\tau),t) - \Delta q^0(A_0(t),t)]\,\mathrm{d}t + \int_{-\infty}^{\infty} [f(\bar{z}(t;\tau),t) - f(A_0(t),t)]\,\mathrm{d}t + \mathrm{O}(\varepsilon).$$

Note that it is here where we have used the fact that the unperturbed solution is homoclinic. Without this assumption, additional terms would appear. The shifted autonomous assumption 1 is now used to convert the arguments in the integrands of the above expression to the (ξ,η) variables as defined in assumption 1. Also,

$$(\bar{z}(t;\tau),t) \longrightarrow (\bar{\xi}(t-\tau),\bar{\eta}(t-\tau),t)$$

where $(\bar{\xi}(t),\bar{\eta}(t))$ is the parametrization of the heteroclinic orbit in the shifted phase space, see (5.2). However, the explicit time dependence disappears since Δq^0 is autonomous in the new variables. Moreover, the Laplacian is invariant under the shift, and we obtain the surprisingly simple expression

$$M(\tau,\varepsilon) = \int_{-\infty}^{\infty} [\Delta Q^0(\bar{\xi}(t),\bar{\eta}(t)) - \Delta Q^0(\xi_A,\eta_A)]\,\mathrm{d}t + \int_{-\infty}^{\infty} [F(\bar{\xi}(t),\bar{\eta}(t),t+\tau) - F(\xi_A,\eta_A,t+\tau)]\,\mathrm{d}t + \mathrm{O}(\varepsilon)$$

by shifting the integration variable. Note from corollary 1 that the distance function has the form

$$d(\tau,\varepsilon) = \varepsilon M(\tau,\varepsilon) + \mathrm{O}(\varepsilon^2)$$

which yields

$$d(\tau,\varepsilon) = \varepsilon\left(\int_{-\infty}^{\infty} [\Delta Q^0(\bar{\xi}(t),\bar{\eta}(t)) - \Delta Q^0(\xi_A,\eta_A)]\,\mathrm{d}t + \int_{-\infty}^{\infty} [F(\bar{\xi}(t),\bar{\eta}(t),t+\tau) - F(\xi_A,\eta_A,t+\tau)]\,\mathrm{d}t\right) + \mathrm{O}(\varepsilon^2)$$

as required. □

Theorem 3 derives a powerful expression for the leading-order term of the distance function associated with viscosity-induced perturbations. We reiterate that most surprising is the fact that the leading-order behaviour is known independently of the perturbed streamfunction. Finally, we write the distance function in the original coordinates.

Corollary 2. *Under the assumptions of theorem 3, the distance function in original coordinates is given by*

$$d(\tau,\varepsilon) = \varepsilon\Bigg(\int_{-\infty}^{\infty} [\Delta q^0(\bar{\xi}(t)+c(t+\tau),\bar{\eta}(t),t+\tau) - \Delta q^0(\xi_A+c(t+\tau),\eta_A,t+\tau)]\,\mathrm{d}t$$

$$+\int_{-\infty}^{\infty}[f(\bar{\xi}(t)+c(t+\tau),\bar{\eta}(t),t+\tau)-f(\xi_A+c(t+\tau),\eta_A,t+\tau)]\,\mathrm{d}t\Big)$$
$$+\mathrm{O}(\varepsilon^2).$$

7. Perturbed flow field models

In this section, we investigate perturbed streamfunctions. It is natural to expect that the perturbed streamfunction will be periodic in time whenever the forcing term F is. Indeed, a steady velocity field is an equilibrium of the underlying PDE. If the equilibrium is hyperbolic, then the perturbed streamfunction would be periodic in time provided the forcing is. It turns out, however, that hyperbolicity fails in the present setting. In the set-up of section 7.1, we derive necessary conditions for periodicity of the perturbed streamfunction. Also, we investigate vorticity-conserving solutions arising in the literature. It is shown that there are specific choices of the forcing term $f(x, y, t)$ for which the perturbed streamfunction is bounded uniformly in time.

Throughout this section, we assume that assumption 1 is met and use shifted coordinates (ξ, η). The perturbed streamfunction $\Psi(\xi, \eta, t; \varepsilon)$ then satisfies

$$\frac{\partial}{\partial t}\Delta\Psi+\{\Psi,\Delta\Psi\}+\beta\Psi_\xi-c\Delta\Psi_\xi=\varepsilon(\Delta^2\Psi+F) \tag{7.1}$$

for $(\xi, \eta) \in \Omega$. Writing

$$\Psi(\xi,\eta,t;\varepsilon)=\Psi^0(\xi,\eta)+\varepsilon\Psi^1(\xi,\eta,t;\varepsilon) \tag{7.2}$$

we obtain

$$\frac{\partial}{\partial t}\Delta\Psi^1+L_0\Psi^1=\Delta^2\Psi^0+F+\varepsilon(\Delta^2\Psi^1-\{\Psi^1,\Delta\Psi^1\}) \tag{7.3}$$

exploiting the fact that Ψ^0 is an equilibrium of (7.1) for $\varepsilon = 0$, that is

$$\{\Psi^0,\Delta\Psi^0\}+\beta\Psi^0_\xi-c\Delta\Psi^0_\xi=0. \tag{7.4}$$

In addition, we have used the definition

$$L_0\Psi^1:=\{\Psi^0,\Delta\Psi^1\}+\{\Psi^1,\Delta\Psi^0\}+\beta\Psi^1_\xi-c\Delta\Psi^1_\xi.$$

7.1. Periodicity of the perturbed streamfunction

As in section 4, we assume that the two-dimensional domain Ω has no boundary. To be more specific, we assume that Ω is given by $\mathbb{R}^2$, $S^1\times\mathbb{R}$, or $S^1\times S^1$. Based on the results stated in section 4, we may assume that Ψ^0 and Ψ^1 are contained in $H^4(\Omega)$. In addition, suppose that $\Delta\Psi^0$ does not vanish identically.

Observe that the operator L_0 has two zero eigenvalues with associated eigenfunctions given by Ψ^0_ξ and Ψ^0_η, respectively, on account of translational invariance. We denote the corresponding eigenfunctions of the adjoint operator L_0^* by Ψ^*_E and Ψ^*_N for eastward and northward translation, respectively. Denoting the L^2-scalar product and L^2-norm by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively, a straightforward calculation shows that

$$\langle\Delta\Psi^0,L_0\Psi^1\rangle=0 \tag{7.5}$$

for all $\Psi^1\in H^3(\Omega)$ as boundary terms do not arise when integrating by parts. Since

$$\langle\Delta\Psi^0,\Psi^0_\xi\rangle=\langle\Delta\Psi^0,\Psi^0_\eta\rangle=0$$

the eigenfunction $\Delta\Psi^0$ of L_0^* is linearly independent of Ψ^*_E and Ψ^*_N. Therefore, zero is an eigenvalue of L_0 (and L_0^*) with geometric multiplicity at least three.

Assumption 4. *Suppose that the wind forcing $F(\xi, \eta, t)$ is periodic in t with period $p > 0$.*

Under this assumption, we will derive conditions that are necessary for the perturbed streamfunction $\Psi(\xi, \eta, t)$ to be periodic.

Proposition 1. *Assume that assumption 4 is met. If $\Psi(\xi, \eta, t)$ is periodic in t with period mp for some $m \in \mathbb{N}$, then the following equalities hold*

$$\left\langle \Psi^0, \frac{1}{p}\int_0^p F(t)\,\mathrm{d}t \right\rangle + \|\Delta\Psi^0\|^2 = 0$$
$$\left\langle \Psi^0, \frac{1}{p}\int_0^p \Delta F(t)\,\mathrm{d}t \right\rangle + \|\nabla\Delta\Psi^0\|^2 = 0$$
$$\left\langle \Psi_E^*, \frac{1}{p}\int_0^p F(t)\,\mathrm{d}t \right\rangle + \langle \Delta\Psi_E^*, \Delta\Psi^0\rangle = 0$$
$$\left\langle \Psi_N^*, \frac{1}{p}\int_0^p F(t)\,\mathrm{d}t \right\rangle + \langle \Delta\Psi_N^*, \Delta\Psi^0\rangle = 0.$$

In particular, on account of the second identity, $\Psi(\xi, \eta, t)$ cannot be periodic in t with period mp whenever F is independent of (ξ, η) and $\Delta\Psi^0$ is not a constant function.

The first identity appearing in proposition 1 is a consequence of conservation of potential vorticity for the unperturbed streamfunction. The remaining three conditions are first-order expansions of (7.3) in the centre subspace spanned by the three known eigenfunctions of L_0 associated with the zero eigenvalue. As a consequence, periodicity of Ψ requires that F is contained in a codimension-four subspace of $H^2(\Omega \times \mathbb{R})$. Note that, if the last two equations do not hold, drifting of the solution in the translational directions is expected.

Proof. Taking the scalar product of (7.1) with Ψ, we obtain

$$\frac{\partial}{\partial t}\|\nabla\Psi\|^2 = -2\varepsilon(\|\Delta\Psi\|^2 + \langle F, \Psi\rangle). \tag{7.6}$$

Indeed, the term

$$\langle \{\Psi, \Delta\Psi\} + \beta\Psi_\xi - c\Delta\Psi_\xi, \Psi\rangle$$

vanishes identically for any function Ψ:

$$\begin{aligned}\langle\{\Psi, \Delta\Psi\}, \Psi\rangle &= \int_\Omega (\Psi_\eta\Delta\Psi_\xi\Psi - \Psi_\xi\Delta\Psi_\eta\Psi)\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &= -\int_\Omega (\Psi(\Delta\Psi_\xi\Psi)_\eta - \Psi(\Delta\Psi_\eta\Psi)_\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &= -\int_\Omega (\Psi\Delta\Psi_\xi\Psi_\eta - \Psi\Delta\Psi_\eta\Psi_\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi = -\langle\{\Psi, \Delta\Psi\}, \Psi\rangle.\end{aligned}$$

Here, we used the fact that boundary terms do not occur when integrating by parts. A similar argument works for the other two terms.

Thus, by applying the integral operator $\int_0^{mp}\mathrm{d}t$ to (7.6) and substituting (7.2),

$$\begin{aligned}0 = -2\varepsilon\bigg(&mp\|\Delta\Psi^0\|^2 + 2\varepsilon\left\langle\Delta\Psi^0, \int_0^{mp}\Delta\Psi^1(t)\,\mathrm{d}t\right\rangle + \varepsilon^2\int_0^{mp}\|\Delta\Psi^1(t)\|^2\,\mathrm{d}t \\ &+\left\langle\Psi^0, \int_0^{mp}F(t)\,\mathrm{d}t\right\rangle + \varepsilon\int_0^{mp}\langle\Psi^1(t), F(t)\rangle\,\mathrm{d}t\bigg)\end{aligned}$$

since the left-hand side is zero because $\Psi(\xi, \eta, t)$ is assumed to be periodic in t. Dividing by ε and using boundedness of Ψ^1 in ε, the first condition follows by setting $\varepsilon = 0$.

The other equalities can be similarly inferred by taking the scalar product of (7.3) with $\Delta\Psi^0$, Ψ_E^*, and Ψ_N^*, respectively, and using the fact that these functions are eigenfunctions of L_0^* with eigenvalue zero. Thus, the terms involving L_0 disappear again. □

7.2. *Concrete models*

Here, we will comment on a class of solutions of the inviscid, unforced equations and their implications for chaotic transport after adding viscous dissipation. We will concentrate on the class of models in which the streamfunction and its Laplacian are linearly related, see (7.7) below. Note that all known closed-form solutions are formed by using this ansatz, see [1] for a discussion. We point out that some of the models are posed on domains Ω possessing non-empty boundaries. Others have streamfunctions which do not belong to $L^2(\Omega)$. Therefore, the results of sections 4 and 7.1 do not necessarily apply. The main issue is then to calculate the perturbed streamfunction and to verify the assumptions made in section 5.

We seek bounded solutions of (7.3), that is

$$\frac{\partial}{\partial t}\Delta\Psi^1 + L_0\Psi^1 = \Delta^2\Psi^0 + F + \varepsilon(\Delta^2\Psi^1 - \{\Psi^1, \Delta\Psi^1\})$$

for small ε. A common feature of the models developed and collected in [1] is that they obey

$$\beta\Psi^0 = c\Delta\Psi^0. \tag{7.7}$$

In other words, Ψ^0 satisfies both terms arising in the unperturbed equilibrium equation (7.4) separately. Note that (7.7) implies that $Q^0(\xi,\eta)$ and the Hamiltonian $\Psi^0(\xi,\eta) + c\eta$ of the Lagrangian flow (5.1) are linearly dependent $Q^0(\xi,\eta) = \frac{\beta}{c}(\Psi^0(\xi,\eta) + c\eta)$. Though the Brown–Samelson theorem is not applicable here, it is clear that chaotic transport is precluded since Ψ^0 does not depend on time.

Condition 1. *The potential vorticity Ψ^0 in shifted coordinates obeys (7.7).*

We remark that if $\partial\Omega$ were empty and Ψ^0 were in $H^4(\Omega)$, (7.7) would imply that β/c is negative. Indeed, taking the scalar product of (7.7) with Ψ^0 and integrating by parts yields $\|\Psi^0\|^2 = -\frac{c}{\beta}\|\nabla\Psi^0\|$.

Before we discuss the consequences of condition 1, we shall give one simple example of a vorticity-conserving streamfunction satisfying condition 1. As mentioned earlier, other solutions can be found in [1]. The example was introduced by Pierrehumbert [23], who used it as a base flow for a formal perturbation analysis:

$$\psi^0(x,y,t) = b\sin(k(x-ct))\sin(ly) \qquad (x,y)\in\mathbb{R}\times S^1.$$

If the wavespeed $c = -\frac{\beta}{k^2+l^2}$ is chosen, condition 1 is indeed met. The streamfunction in the moving frame is given by

$$\Psi^0(\xi,\eta) = b\sin(k\xi)\sin(l\eta) \qquad (\xi,\eta)\in\mathbb{R}\times S^1. \tag{7.8}$$

It is straightforward to see that equation (5.1) describing Lagrangian trajectories has cat's eyes whenever

$$b > \frac{\beta}{k(k^2+l^2)}$$

see figure 4 for the level curves of the Hamiltonian associated with Pierrehumbert's streamfunction.

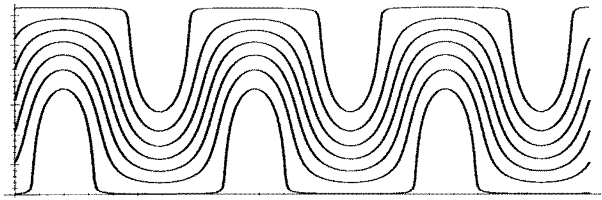

Figure 4. Level curves of the Hamiltonian $\Psi^0(\xi,\eta)+c\eta$ with Ψ^0 given in (7.8). Here, $b = 0.4$, $\beta = 1$, $k = 2$ and $l = 1$.

Returning to the general case, as a consequence of condition 1, $r\Psi^0$ is an equilibrium of the Eulerian equation, that is, it satisfies (7.4) for all $r \in \mathbb{R}$. This suggests the ansatz

$$\Psi^1 = r\Psi^0 + \Psi^2. \tag{7.9}$$

Using (7.7), it is straightforward to calculate that (7.3) is equivalent to

$$\frac{\beta}{c}\frac{\partial r}{\partial t}\Psi^0 + \frac{\partial}{\partial t}\Delta\Psi^2 + L_0\Psi^2 = \frac{\beta^2}{c^2}(1+\varepsilon r)\Psi^0 + F + \varepsilon(\Delta^2\Psi^2 - \{\Psi^2, \Delta\Psi^2\}). \tag{7.10}$$

It would be difficult to solve equation (7.10) for general forcing terms $F(\xi,\eta,t)$. We therefore restrict ourselves to forcing terms of the form

$$F(\xi,\eta,t) = A(t)\Psi^0(\xi,\eta) \tag{7.11}$$

where $A(t)$ is bounded. Substituting this expression and setting $\Psi^2 = 0$, we see that (7.10) is equivalent to

$$\frac{\beta}{c}\frac{\partial r}{\partial t}\Psi^0 = \frac{\beta^2}{c^2}(1+\varepsilon r)\Psi^0 + A(t)\Psi^0$$

and it suffices to solve

$$\frac{\partial r}{\partial t} = \frac{\beta}{c}\varepsilon r + \frac{\beta}{c} + \frac{c}{\beta}A(t). \tag{7.12}$$

We then have the explicit general solution

$$r(t;\varepsilon) = e^{\beta\varepsilon t/c}r_0 + \int_0^t e^{\beta\varepsilon(t-s)/c}\left(\frac{\beta}{c} + \frac{c}{\beta}A(s)\right)\mathrm{d}s \tag{7.13}$$

of (7.12). Multiplying with $e^{-\beta\varepsilon t/c}$, we obtain

$$e^{-\beta\varepsilon t/c}r(t;\varepsilon) = r_0 + \int_0^t e^{-\beta\varepsilon s/c}\left(\frac{\beta}{c} + \frac{c}{\beta}A(s)\right)\mathrm{d}s. \tag{7.14}$$

For $\beta/c < 0$, the solution $r(t;\varepsilon)$ is therefore bounded for $t \to -\infty$ if and only if the limit of the integral term on the right-hand side of (7.14) exists for $t \to -\infty$. Setting $\varepsilon = 0$, we see that $r(t;0)$ is bounded for $t \to \infty$ if and only if

$$\int_0^t \left(\frac{\beta}{c} + \frac{c}{\beta}A(s)\right)\mathrm{d}s \tag{7.15}$$

is bounded as $t \to \infty$. Therefore, we impose the following condition on the forcing term $F(\xi,\eta,t)$.

Condition 2. *The forcing term satisfies*

$$F(\xi,\eta,t) = \left(-\frac{\beta^2}{c^2} + a(t)\right)\Psi^0(\xi,\eta)$$

for some smooth function $a(t)$. Moreover, there exist constants $\delta > 0$, $T \geqslant 0$ and $K, p \in \mathbb{R}$ such that the following holds.

(i) For $t \geqslant -T$, the amplitude $a(t)$ is periodic in t with period p and has mean zero, that is, $a(t+p) = a(t)$ for all $t \geqslant -T$, and $\int_{-T}^{-T+p} a(t)\,\mathrm{d}t = 0$.

(ii) For $t \leqslant -T$, the amplitude $a(t)$ decays exponentially, that is, $|a(t)| \leqslant K\mathrm{e}^{\delta t}$ for $t \leqslant -T$.

We then state the following proposition.

Proposition 2. *Suppose that conditions 1 and 2 are met. In addition, we assume that $\frac{\beta}{c} < 0$. The perturbed streamfunction $\Psi(t;\varepsilon)$ given by*

$$\Psi(\xi,\eta,t;\varepsilon) = \left(1 + \varepsilon\frac{c}{\beta}\int_{-\infty}^{t} \mathrm{e}^{\beta\varepsilon(t-s)/c}\, a(s)\,\mathrm{d}s\right)\Psi^0(\xi,\eta)$$

is then bounded uniformly in $t \in \mathbb{R}$ and $\varepsilon \in [0,\varepsilon_0]$, and satisfies (7.1). Moreover, there exists a function $F(t;\varepsilon)$ periodic in t with period p such that

$$|\Psi(\xi,\eta,t;\varepsilon) - (1+\varepsilon F(t;\varepsilon))\Psi^0(\xi,\eta)| \leqslant C\mathrm{e}^{\beta\varepsilon t/c}$$

for some constant C and $t \geqslant -T$. In other words, for $\varepsilon > 0$, the perturbed streamfunction is asymptotically periodic in time.

Proof. Comparing (7.11) with condition 2, we have $A(t) = -\frac{\beta^2}{c^2} + a(t)$. Formula (7.13) for $r(t;\varepsilon)$ then reads

$$r(t;\varepsilon) = \mathrm{e}^{\beta\varepsilon t/c} r_0 + \frac{c}{\beta}\int_0^t \mathrm{e}^{\beta\varepsilon(t-s)/c} a(s)\,\mathrm{d}s.$$

Observe that

$$r_0 := \frac{c}{\beta}\int_{-\infty}^{0} \mathrm{e}^{-\beta\varepsilon s/c} a(s)\,\mathrm{d}s$$

is well defined since $a(t)$ decays exponentially for $t \to -\infty$ by condition 2. We then have

$$r(t;\varepsilon) = \frac{c}{\beta}\int_{-\infty}^{t} \mathrm{e}^{\beta\varepsilon(t-s)/c} a(s)\,\mathrm{d}s$$

and it remains to show that $r(t;\varepsilon)$ is bounded for $t \to \infty$. Since $\frac{\beta}{c} < 0$, it suffices to consider the integral term

$$\mathrm{e}^{\beta\varepsilon t/c}\int_0^t \mathrm{e}^{-\beta\varepsilon s/c} a(s)\,\mathrm{d}s.$$

Since $a(t)$ is periodic in t for $t \geqslant 0$ and has mean zero, we may expand it in a Fourier series

$$a(t) = \sum_{n=1}^{\infty} (a_n \sin(2\pi nt/p) + b_n \cos(2\pi nt/p)).$$

Using the formula

$$e^{\beta\varepsilon t/c}\int_0^t e^{-\beta\varepsilon s/c}\sin(2\pi ns/p)\,\mathrm{d}s = -\left(\frac{\beta^2\varepsilon^2}{c^2}+\frac{4\pi^2 n^2}{p^2}\right)^{-1} \times\left(\frac{\beta\varepsilon}{c}\sin(2\pi nt/p)+\frac{2\pi n}{p}\cos(2\pi nt/p)-\frac{2\pi n}{p}e^{\beta\varepsilon t/c}\right)$$

and the analogue for the cosine terms, the statement of the proposition follows. □

Note that the results are still true for $\frac{\beta}{c} > 0$ with t replaced by $-t$. However, the perturbed streamfunction would then be asymptotically periodic for negative times.

Finally, condition 1 and (7.11) allow us to derive an explicit formula for the Melnikov integral M.

Lemma 7. *Assume that assumptions 1, 2 and condition 1 are met. Suppose that the forcing term is given by (7.11), i.e.* $F(\xi,\eta,t) = A(t)\Psi^0(\xi,\eta)$. *The Melnikov function* M *appearing in (6.2) is then given by*

$$M(\tau) = \frac{\beta^2}{c}\int_{-\infty}^{\infty}(\eta_A - \bar{\eta}(t))\,\mathrm{d}t + c\int_{-\infty}^{\infty}A(t+\tau)(\eta_A-\bar{\eta}(t))\,\mathrm{d}t.$$

Proof. Equation (5.1) in shifted coordinates has the first integral $H(\xi,\eta) = \Psi^0(\xi,\eta) + c\eta$. In particular, $\frac{\partial}{\partial t}H(\xi(t),\eta(t)) = 0$ and we obtain

$$\Psi^0(\bar{\xi}(t),\bar{\eta}(t)) - \Psi^0(\xi_A,\eta_A) = c(\eta_A - \bar{\eta}(t))$$

for all t. Therefore,

$$\Delta Q^0(\bar{\xi}(t),\bar{\eta}(t)) - \Delta Q^0(\xi_A,\eta_A) = \frac{\beta^2}{c}(\eta_A - \bar{\eta}(t))$$

using (7.7) and $Q^0 = \Delta\Psi^0 + \beta\eta$, and the first integral of the Melnikov function in (6.2) is equal to

$$\frac{\beta^2}{c}\int_{-\infty}^{\infty}(\eta_A - \bar{\eta}(t))\,\mathrm{d}t.$$

This result can be confirmed by calculating the Melnikov integral for $F(t) \equiv 0$ using the explicit expression for the perturbed streamfunction Ψ provided in proposition 2. The second integral is computed in a similar fashion. □

Finally, we note that the results stated above remain valid for two-dimensional incompressible, vorticity-preserving flows, that is for $\beta = c = 0$, provided $\Psi^0 = \Delta\Psi^0$ is met. Essentially, the fractions β/c are replaced by 1 in the proofs. As a consequence, we emphasize that the first term in the Melnikov integral must vanish. Indeed, the calculations in the proof of lemma 7 show that the first integrand

$$\Delta Q^0(\bar{\xi}(t),\bar{\eta}(t)) - \Delta Q^0(\xi_A,\eta_A) = \Psi^0(\bar{\xi}(t),\bar{\eta}(t)) - \Psi^0(\xi_A,\eta_A) = H(\bar{\xi}(t),\bar{\eta}(t)) - H(\xi_A,\eta_A) = 0$$

vanishes.

8. Transport after perturbation

As discussed in the introduction, the aim of this paper is to study the nature of transport between the various component parts of a large-scale fluid structure such as a meandering ocean jet. A key role is played by the vortical structures that flank the jet, namely the so-called cat's eyes. Of interest then is the transport of fluid from the jet to the cat's eye and from the cat's eye to the ambient, retrograde fluid. Of specific interest is what physical mechanisms might act as facilitators of such transport and whether this transport will have a chaotic nature. In this work we have added to the equation of potential vorticity a term reflecting the dissipative effect of viscosity and a forcing that might crudely be viewed as wind forcing on the surface of the ocean.

We then take a model for the inviscid fluid which is a wave travelling in an easterly direction with a meandering structure. We assume that this base wave is steady in the moving frame and is, moreover, periodic in the easterly direction. A cat's eye flanking the meandering jet is identified, in such a model, with a heteroclinic loop which can, in turn, be viewed as a homoclinic orbit if the periodicity in x is exploited and the problem cast on a cylinder. The question is whether these homoclinic orbits split under the effect of the perturbing terms, viscosity and forcing, introduced into the PDE.

To answer this question, under the assumption that the perturbed flow field satisfies an appropriate boundedness assumption 3, we have derived the explicit expression

$$M(\tau) = \int_{-\infty}^{\infty} [\Delta Q^0(\bar{\xi}(t), \bar{\eta}(t)) - \Delta Q^0(\xi_A, \eta_A)]\,\mathrm{d}t + \int_{-\infty}^{\infty} [F(\bar{\xi}(t), \bar{\eta}(t), t+\tau) - F(\xi_A, \eta_A, t+\tau)]\,\mathrm{d}t \qquad (8.1)$$

for the Melnikov function that represents the first-order term in the expansion of the distance function $d(\tau, \varepsilon) = \varepsilon M(\tau) + \mathrm{O}(\varepsilon^2)$, see theorem 3. The surprising fact that emerges is the independence of this expression from the perturbed flow field. This is an incredibly useful feature of the calculation as the perturbed flow field is unknown.

A central assumption in our analysis is the boundedness assumption 3. We have derived some conditions under which this will hold as a consequence of the perturbed flow field being periodic and we have also given a specific example under which it is satisfied. However, in general we cannot expect it to be met, see below, and in a further paper we shall consider cases under which it fails.

In the following, we shall discuss the implications of our Melnikov analysis for various different types of forcing functions. Some surprising conclusions can be made about the nature of the transport in each of these cases.

8.1. Spatially independent wind forcing

First, we assume that the forcing function does not depend on the spatial variables. In other words, suppose that $f = f(t)$ does not depend on x and y. It is then clear that it is also independent of ξ and η in the moving frame, i.e. $F = F(t)$. Under this condition, the calculation of the Melnikov function can be considerably simplified as the second integral in (8.1) is identically zero.

The resulting distance function has then the form

$$d(\tau, \varepsilon) = \varepsilon \int_{-\infty}^{\infty} [\Delta Q^0(\bar{\xi}(t), \bar{\eta}(t)) - \Delta Q^0(\xi_A, \eta_A)]\,\mathrm{d}t + \mathrm{O}(\varepsilon^2). \qquad (8.2)$$

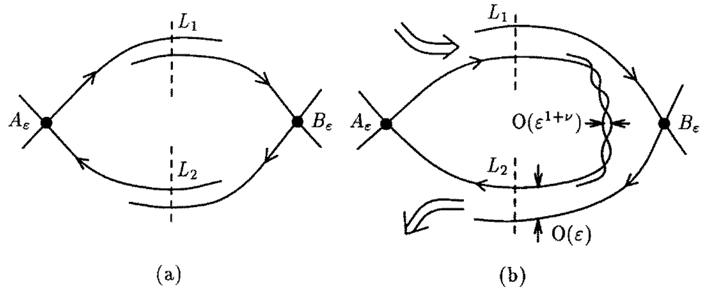

Figure 5. A perturbed cat's eye. In the context of incompressible fluids, (*b*) must occur leading to an avenue along which fluid parcels pass from the northern to the southern part, or vice versa.

If the integral in the first-order term, i.e. the Melnikov function, is non-zero then there is a striking implication for the separation of stable and unstable manifolds. Indeed, they must separate a uniform distance apart, up to first order, independently of the time slice. Thus, in this situation, there can be *no* intersection of the relevant manifolds under a viscous perturbation consistent with (4.1) for small enough ε. Moreover, it is a consequence of lemma 7 that, for the specific models we have considered, the Melnikov function is given by

$$\frac{\beta^2}{c}\int_{-\infty}^{\infty}(\eta_A - \bar{\eta}(t))\,\mathrm{d}t$$

which is likely to be non-zero if $\beta \neq 0$. In case $\beta = 0$, the non-oceanographic case, we cannot conclude that intersections are forbidden, but they must happen at higher order.

Suppose now that there is a cat's eye in the unperturbed flow. We assume that one of the Melnikov functions is non-zero. Both the lower and upper heteroclinics will then split. Indeed, by lemma 5, the sum of the Melnikov integrals M_1 and M_2 associated with the two separatrices forming the cat's eye is of order ε. Therefore, $M_1 = -M_2$ up to order ε, and, in particular, if one of the integrals is non-zero, so is the other. There are now two possibilities of how the separatrices can split, see figure 5. As discussed in section 3, the case depicted in figure 5(*a*) is impossible for incompressible fluids due to area conservation. Indeed, $M_1 = -M_2$ up to order ε, and the separatrices therefore break in different directions. The splitting of the manifolds must then occur in the manner depicted in figure 5(*b*). The stable and unstable manifolds of the point A_ε may still intersect and, also due to area conservation, in fact must intersect. It is a consequence of lemma 5 and theorem 2 that the splitting distance between the manifolds is of higher order, in fact $O(\varepsilon^{1+\nu})$ for some $\nu > 0$. Indeed, both results are applicable since hypothesis 5 is a consequence of theorem 3, while hypothesis 3 is met with $h^0(\xi,\eta) = \Psi^0(\xi,\eta) + c\eta$ and $h^1(\xi,\eta,t;\varepsilon) = \Psi^1(\xi,\eta,t;\varepsilon)$.

The picture one gets here is then of the possibility of the transport of fluid between different regimes by virtue of a channel opening up, as depicted in figure 5(*b*) for the north to south case. Since the heteroclinics split at a lower order than the inner homoclinic, the probability is great of a fluid particle being carried past the vortex region forming the cat's eye, rather than be entrained into it. In this situation therefore chaotic transport is severely inhibited. However, an avenue is opened up for fluid to escape from one region to another in a non-chaotic fashion. It is feasible that

$$\left|\int_{-\infty}^{\infty}[\Delta Q^0(\bar{\xi}(t),\bar{\eta}(t)) - \Delta Q^0(\xi_A,\eta_A)]\,\mathrm{d}t\right| > 0$$

for almost all non-trivial unperturbed flows, and hence the indications are that, in general, chaotic mixing will not result from including viscous effects. This is unexpected since the manifolds are known to exhibit tangling under almost any perturbation. For a model to predict chaotic transport, it is therefore necessary that the inviscid flow disobey either the shifted autonomous or the homoclinic assumption.

8.2. Meridional wind forcing

Next, we suppose that the wind forcing depends only on the meridional variable. In other words, we set $f = f(y)$, whence $F(\eta) = f(\eta)$ in the moving frame. In this case the second integral in (8.1) does contribute to the Melnikov function, but remains constant. Indeed,

$$M(\tau) = \int_{-\infty}^{\infty} [\Delta Q^0(\bar{\xi}(t), \bar{\eta}(t)) - \Delta Q^0(\xi_A, \eta_A)]\,\mathrm{d}t + \int_{-\infty}^{\infty} [f(\bar{\eta}(t)) - f(\eta_A)]\,\mathrm{d}t$$

which is clearly independent of τ. Here, we have replaced the function $F(\eta)$ by $f(\eta)$ in the second integrand.

One can imagine choosing f in different ways that would produce a Melnikov function which is either positive or negative (or zero). But, in any case, we would not have transverse intersections of stable and unstable manifolds and thus the chaotic nature of the transport would be inhibited as above.

8.3. Temporally independent wind forcing

If the wind forcing depends only on the spatial variables x and y, i.e. $f = f(x, y)$, then in the moving frame the forcing becomes dependent on time through its dependence on x. Indeed,

$$F(\xi, \eta, t) = f(\xi + ct, \eta).$$

If we assume that the forcing f is periodic in x, which we actually have to do in order to satisfy our hypotheses, then F is periodic in t. Replacing F again by f in the second integral in (8.1), it follows that the Melnikov function

$$\int_{-\infty}^{\infty} [\Delta Q^0(\bar{\xi}(t), \bar{\eta}(t)) - \Delta Q^0(\xi_A, \eta_A)]\,\mathrm{d}t + \int_{-\infty}^{\infty} [f(\bar{\xi}(t) + c(t+\tau), \bar{\eta}(t)) - f(\xi_A + c(t+\tau), \eta_A)]\,\mathrm{d}t$$

is also periodic in τ. Note that this periodicity holds even if the underlying flow field is not periodic. It follows that if the Melnikov function has one zero then it has infinitely many zeros. Moreover, it is not hard to concoct forcing functions f which render a zero of the Melnikov integral. This case would then naturally lead to the occurrence of complicated heteroclinic tangling.

8.4. General forcing

For general forcing functions the calculation of the Melnikov integral still holds as in (8.1), but conclusions may be harder to make. If, however, the forcing $f(x, y, t)$ enjoys some periodicity in both x and t then the forcing function in a moving frame

$$F(\xi, \eta, t) = f(\xi + ct, \eta, t)$$

will be quasiperiodic in t. It follows again that the Melnikov function will have the same property, namely be quasiperiodic in τ. As before, if it has one zero, it will have infinitely many.

8.5. Perturbations without forcing

We close this section with a discussion of the simplest case, namely whether there is no forcing at all. In other words, we assume $f = 0$. This case falls under all of the above but the conclusions do not apply as a basic hypothesis is not satisfied. Indeed, it follows from equation (7.6) that the quantity

$$\iint |\nabla\psi(x, y, t; \varepsilon)|^2 \, \mathrm{d}x \, \mathrm{d}y$$

will decay to zero for $\varepsilon > 0$ as $t \to \infty$. But then it would be impossible to have a field which is close to the $\varepsilon = 0$ flow field for all time when ε is non-zero. Thus, the boundedness hypothesis does not hold.

It should be commented that the case of no forcing is that studied in [25] and there stable and unstable manifolds are found numerically to have many intersections. It would be tempting to think that the boundedness hypothesis not being satisfied supplied an explanation for this discrepancy between the results of this paper, at least extrapolated to the case of a decaying streamfunction, and those of [25]. However, in a further paper we show that the case of an unbounded streamfunction is covered by our theory, provided the streamfunctions stay close for long enough. An explanation must therefore be sought elsewhere and further discussion of this point will appear in this forthcoming paper.

9. Conclusions

We have considered the effect of viscosity and forcing on the Lagrangian transport of fluid parcels. The set-up we adopted was of an inviscid, incompressible two-dimensional velocity field that is steady in a moving frame. The physical effects of viscosity and forcing are then added to the vorticity equation to produce a perturbed, unsteady velocity field.

The nature of the transport depends crucially on the type of forcing. For natural examples of forcing (spatially independent or only meridionally dependent) the separatrices of a steady, in a moving frame, velocity field have been shown to open up for the perturbed (unsteady) field to produce a channel through which fluid can be transported across a cat's eye without being entrained into the vortical regime. For other types of forcing, for instance periodic in the horizontal direction, transverse intersections of stable and unstable manifolds results to give a heteroclinic tangle and associated complex transport.

The subtlety of the results lies in the fact that the velocity fields are not explicitly known, but only implicitly through the PDEs they satisfy. We have shown that a periodic, in time, velocity field cannot be expected and thus the theory has been developed under the assumption that the perturbed velocity field is only bounded.

The mathematical development involved a new extension for the Melnikov theory to weak, non-smooth (in the parameter ϵ) perturbations. The Melnikov calculation renders a surprisingly simple formula. A key point is that this formula does not involve the perturbed velocity field. While, in general, it would be expected that the Melnikov function does not depend on the perturbed trajectories, it usually does depend on the perturbed velocity field. The effects of viscosity and forcing are, however, of such a form that we are saved from this complication and the Melnikov function can be calculated knowing only information from the $\epsilon = 0$ velocity field.

From the oceanographic point of view, this analysis offers some very suggestive conclusions. The Gulf Stream is perhaps the best known example of a meandering jet and, based on models of perturbed jets, see for instance [4, 6], it is accepted that cat's eye vortical regions lie in the troughs and under the crests of the meanders. If one accepts a barotropic model of the Gulf Stream then the analysis of this paper suggests that viscosity, here in the sense of an eddy viscosity, and westerly winds would tend to promote the direct ejection of fluid parcels from the centre of the jet to the ambient waters without their being entrained, even temporarily, in the vortical regions. This is in contrast to the kinematic and perturbed jet models, see [6, 11, 14, 23, 24, 26, 27], which predict a predominance of transport between the vortical regions and the jet and ambient water separately. The immediate ejection of fluid parcels is, in these models, possible but unlikely. The real Gulf Stream, as well as any other ocean jet, is obviously an extremely complex structure which exhibits all of the above possibilities. However, the analysis shows that the pure effects of viscosity and forcing have a certain unanticipated effect which may well lead to an unexpectedly large occurrence of immediate ejection of fluid parcels.

Acknowledgments

SB and CKRTJ were partially supported by the National Science Foundation under grant DMS-94-02774. CKRTJ and BS were partially supported by the Office of Naval Research under grant N00014-93-1-0691. BS was additionally supported by a Feodor–Lynen Fellowship of the Alexander von Humboldt Foundation.

References

[1] Balasuriya H S 1996 Viscosity induced transport in barotropic jets *PhD Thesis* Brown University
[2] Balasuriya H S Vanishing viscosity in the barotropic β-plane *J. Math. Anal. Appl.* **214** 128–50
[3] Behringer R P, Meyers S D and Swinney H L 1990 Chaos and mixing in a geostrophic flow *Phys. Fluids* A **3** 1243–9
[4] Bower A S 1991 A simple kinematic mechanism for mixing fluid parcels across a meandering jet *J. Phys. Oceanography* **21** 173–80
[5] Bower A S and Lozier M S 1994 A closer look at particle exchange in the gulf stream *J. Phys. Oceanography* **24** 1399–418
[6] Bower A S and Rossby T 1989 Evidence of cross-frontal exchange processes in the gulf stream based on isopycnal RAFOS float data *J. Phys. Oceanography* **19** 1177–90
[7] Brown M G and Samelson R M 1994 Particle motion in vorticity-conserving, two-dimensional incompressible flows *Phys. Fluids* **6** 2875–6
[8] Chow S-N and Hale J K 1982 *Methods of Bifurcation Theory* (New York: Springer)
[9] Coppel W A 1978 *Dichotomies in Stability Theory (Lecture Notes in Mathematics 629)* (New York: Springer)
[10] Cushman-Roisin B 1994 *Introduction to Geophysical Fluid Dynamics* (Englewood Cliffs, NJ: Prentice-Hall)
[11] del-Castillo-Negrete D and Morrison P J 1993 Chaotic transport by Rossby waves in shear flow *Phys. Fluids* A **5** 948–65
[12] Deng B 1991 The bifurcations of countable connections from a twisted heteroclinic loop *SIAM J. Math. Anal.* **22** 653–79
[13] Fiedler B and Scheurle J 1996 *Discretization of Homoclinic Orbits, Rapid Forcing and Invisible Chaos (Memoirs of the American Mathematical Society 570)* (Providence, RI: American Mathematical Society)
[14] Knobloch E and Weiss J B 1987 Chaotic advection by modulated travelling waves *Phys. Rev.* A **36** 1522–4
[15] Kokubu H 1988 Homoclinic and heteroclinic bifurcations in vector fields *Japan J. Appl. Math.* **5** 455–501
[16] Ladyzhenskaya O A 1969 *The Mathematical Theory of Viscous Incompressible Flow* (New York: Gordon and Breach)
[17] Lin X-B 1990 Using Melnikov's method to solve Silnikov's problems *Proc. R. Soc.* A **116** 295–325

[18] Lozier M S, Pratt L J, Rogerson A M and Miller P D Exchange geometry revealed by float trajectories in the gulf stream *J. Phys. Oceanography* to appear

[19] Meyers S D 1994 Cross-frontal mixing in a meandering jet *J. Phys. Oceanography* **24** 1641–6

[20] Miller P, Jones C K R T, Haller G and Pratt L J 1996 Chaotic mixing across oceanic jets *Chaotic, Fractal, and Nonlinear Signal Processing (AIP Conf. Proc. 375)* (Woodbury, NY: AIP)

[21] Ottino J M 1990 Mixing, chaotic advection, and turbulence *Ann. Rev. Fluid Mech.* **22** 207–53

[22] Pedlosky J 1979 *Geophysical Fluid Dynamics* (New York: Springer)

[23] Pierrehumbert R T 1991 Chaotic mixing of tracer and vorticity by modulated travelling Rossby waves *Geophys. Astrophys. Fluid Dynam.* **58** 285–319

[24] Pratt L J, Lozier M S and Beliakova N 1995 Parcel trajectories in quasigeostrophic jets: neutral modes *J. Phys. Oceanography* **25** 1451–66

[25] Rogerson A M, Miller P D, Pratt L J, Jones C K R T and Biello J 1996 Chaotic mixing in a barotropic meandering jet *Preprint*

[26] Samelson R M 1992 Fluid exchange across a meandering jet *J. Phys. Oceanography* **22** 431–40

[27] Samelson R M Chaotic transport by mesoscale motions *Stochastic Modelling and Physical Oceanography* ed R Adler *et al* (Boston, MA: Birkhauser)

[28] Sandstede B 1993 Verzweigungstheorie homokliner Verdopplungen *Doctoral Thesis* University of Stuttgart

[29] Sandstede B, Jones C K R T and Alexander J C 1997 Existence and stability of N-pulses on optical fibers with phase-sensitive amplifiers *Physica* D **106** 167–206

[30] Temam R 1983 *Navier–Stokes Equations and Nonlinear Functional Analysis (CBMS-NSF Regional Conference Series in Appl. Math.)* (Philadelphia, PA: SIAM)

[31] Weiss J B and Knobloch E 1989 Mass transport and mixing by modulated travelling waves *Phys. Rev.* A **40** 2579–89

[32] Wiggins S 1992 *Chaotic Transport in Dynamical Systems* (New York: Springer)