MATHS IB: Calculus

§1.7 Second-order nonhomogeneous DEs

Sanjeeva Balasuriya

THE UNIVERSITY of ADELAIDE

A linear combination question

Breakout: Suppose you are told that when you form a linear combination of a function $f(t)$ and its derivatives $f'(t)$ and $f''(t)$, you get $-3e^{5t}$. What is the most general $f(t)$ that this could be from?

(a) $A\cos 3t$

(b) Ae^{5t}

(c) Ae^{-3t}

(d) $At^2 + Bt + C$

A linear combination question

Breakout: Suppose you are told that when you form a linear combination of a function $f(t)$ and its derivatives $f'(t)$ and $f''(t)$, you get $-3e^{5t}$. What is the most general $f(t)$ that this could be from?

(a) $A\cos 3t$

(b) Ae^{5t}

(c) Ae^{-3t}

(d) $At^2 + Bt + C$

$$c_1 f(t) + c_2 f'(t) + c_3 f''(t) = -3e^{5t}$$

Finding any old (not general) solution

Example: Find any solution to the *nonhomogeneous* DE

$$y''(t) + 6y'(t) + 25y(t) = 8e^{-7t}$$

3min

Finding any old (not general) solution

Example: Find any solution to the *nonhomogeneous* DE

$$y''(t) + 6y'(t) + 25y(t) = 8e^{-7t}$$

3min

- Using insight from previous question: guess $y(t) = Ae^{-7t}$

Finding any old (not general) solution

Example: Find any solution to the *nonhomogeneous* DE

$$y''(t) + 6y'(t) + 25y(t) = 8e^{-7t}$$

3min

- Using insight from previous question: guess $y(t) = Ae^{-7t}$
- $y'(t) = -7Ae^{-7t}$ and $y''(t) = 49Ae^{-7t}$

Finding any old (not general) solution

Example: Find any solution to the *nonhomogeneous* DE

$$y''(t) + 6y'(t) + 25y(t) = 8e^{-7t}$$

3min

- Using insight from previous question: guess $y(t) = Ae^{-7t}$
- $y'(t) = -7Ae^{-7t}$ and $y''(t) = 49Ae^{-7t}$
- Substitute into DE:

$$49Ae^{-7t} + 6\left(-7Ae^{-7t}\right) + 25Ae^{-7t} = 8e^{-7t}$$

Finding any old (not general) solution

Example: Find any solution to the *nonhomogeneous* DE

$$y''(t) + 6y'(t) + 25y(t) = 8e^{-7t}$$

3min

- Using insight from previous question: guess $y(t) = Ae^{-7t}$
- $y'(t) = -7Ae^{-7t}$ and $y''(t) = 49Ae^{-7t}$
- Substitute into DE:

$$49Ae^{-7t} + 6\left(-7Ae^{-7t}\right) + 25Ae^{-7t} = 8e^{-7t}$$

- $32Ae^{-7t} = 8e^{-7t}$

Finding any old (not general) solution

Example: Find <u>any</u> solution to the *nonhomogeneous* DE

$$y''(t) + 6y'(t) + 25y(t) = 8e^{-7t}$$

3min

- Using insight from previous question: guess $y(t) = Ae^{-7t}$
- $y'(t) = -7Ae^{-7t}$ and $y''(t) = 49Ae^{-7t}$
- Substitute into DE:

$$49Ae^{-7t} + 6\left(-7Ae^{-7t}\right) + 25Ae^{-7t} = 8e^{-7t}$$

- $32Ae^{-7t} = 8e^{-7t} \quad \Rightarrow \quad A = 1/4$

Finding any old (not general) solution

Example: Find any solution to the *nonhomogeneous* DE

$$y''(t) + 6y'(t) + 25y(t) = 8e^{-7t}$$

3min

- Using insight from previous question: guess $y(t) = Ae^{-7t}$
- $y'(t) = -7Ae^{-7t}$ and $y''(t) = 49Ae^{-7t}$
- Substitute into DE:

$$49Ae^{-7t} + 6\left(-7Ae^{-7t}\right) + 25Ae^{-7t} = 8e^{-7t}$$

- $32Ae^{-7t} = 8e^{-7t} \quad \Rightarrow \quad A = 1/4$
- A solution:

$$y(t) = \frac{1}{4}e^{-7t}$$

Finding a particular solution

Example: Find any solution to the *nonhomogeneous* DE

$$y''(t) + 6y'(t) + 25y(t) = 8e^{-7t}$$

3min

- Using insight from previous question: guess $y(t) = Ae^{-7t}$
- $y'(t) = -7Ae^{-7t}$ and $y''(t) = 49Ae^{-7t}$
- Substitute into DE:

$$49Ae^{-7t} + 6\left(-7Ae^{-7t}\right) + 25Ae^{-7t} = 8e^{-7t}$$

- $32Ae^{-7t} = 8e^{-7t} \quad \Rightarrow \quad A = 1/4$
- A *particular* solution:

$$y_p(t) = \frac{1}{4}e^{-7t}$$

Another linear combination question

Breakout: Suppose you are told that when you form a linear combination of a function $f(t)$ and its derivatives $f'(t)$ and $f''(t)$, you get $\cos 3t$. What is the most general $f(t)$ that this could be from?

(a) $A \cos 3t$

(b) $A \sin 3t$

(c) $A \cos \omega t$

(d) $A \cos 3t + B \sin 3t$

Another linear combination question

Breakout: Suppose you are told that when you form a linear combination of a function $f(t)$ and its derivatives $f'(t)$ and $f''(t)$, you get $\cos 3t$. What is the most general $f(t)$ that this could be from?

(a) $A\cos 3t$

(b) $A\sin 3t$

(c) $A\cos\omega t$

(d) $A\cos 3t + B\sin 3t$

$$c_1 f(t) + c_2 f'(t) + c_3 f''(t) = \cos 3t$$

Must choose both sine and cosine since when taking derivatives they 'go to each other'

Guessing the form of a particular solution

Task: to find a *particular* solution $y_p(t)$ to:

$$y''(t) + a\,y'(t) + b\,y(t) = f(t)$$

[Table 1.2]

Form of $f(t)$	Guess for $y_p(t)$
$ce^{\alpha t}$	$Ae^{\alpha t}$

Guessing the form of a particular solution

Task: to find a *particular* solution $y_p(t)$ to:

$$y''(t) + a\,y'(t) + b\,y(t) = f(t)$$

[Table 1.2]

Form of $f(t)$	Guess for $y_p(t)$
$ce^{\alpha t}$	$Ae^{\alpha t}$
$c\cos\omega t$ or $c\sin\omega t$	$A\cos\omega t + B\sin\omega t$

Guessing the form of a particular solution

Task: to find a *particular* solution $y_p(t)$ to:

$$y''(t) + a\,y'(t) + b\,y(t) = f(t)$$

[Table 1.2]

Form of $f(t)$	Guess for $y_p(t)$
$ce^{\alpha t}$	$Ae^{\alpha t}$
$c\cos\omega t$ or $c\sin\omega t$	$A\cos\omega t + B\sin\omega t$
$ce^{\alpha t}\cos\omega t$ or $ce^{\alpha t}\sin\omega t$	

Guessing the form of a particular solution

Task: to find a *particular* solution $y_p(t)$ to:

$$y''(t) + a\,y'(t) + b\,y(t) = f(t)$$

[Table 1.2]

Form of $f(t)$	Guess for $y_p(t)$
$ce^{\alpha t}$	$Ae^{\alpha t}$
$c\cos\omega t$ or $c\sin\omega t$	$A\cos\omega t + B\sin\omega t$
$ce^{\alpha t}\cos\omega t$ or $ce^{\alpha t}\sin\omega t$	$Ae^{\alpha t}\cos\omega t + Be^{\alpha t}\sin\omega t$

Guessing the form of a particular solution

Task: to find a *particular* solution $y_p(t)$ to:

$$y''(t) + a\,y'(t) + b\,y(t) = f(t)$$

[Table 1.2]

Form of $f(t)$	Guess for $y_p(t)$
$ce^{\alpha t}$	$Ae^{\alpha t}$
$c\cos\omega t$ or $c\sin\omega t$	$A\cos\omega t + B\sin\omega t$
$ce^{\alpha t}\cos\omega t$ or $ce^{\alpha t}\sin\omega t$	$Ae^{\alpha t}\cos\omega t + Be^{\alpha t}\sin\omega t$
$a_0 + a_1 t + \cdots + a_n t^n$	

Guessing the form of a particular solution

Task: to find a *particular* solution $y_p(t)$ to:

$$y''(t) + a\,y'(t) + b\,y(t) = f(t)$$

[Table 1.2]

Form of $f(t)$	Guess for $y_p(t)$
$ce^{\alpha t}$	$Ae^{\alpha t}$
$c\cos\omega t$ or $c\sin\omega t$	$A\cos\omega t + B\sin\omega t$
$ce^{\alpha t}\cos\omega t$ or $ce^{\alpha t}\sin\omega t$	$Ae^{\alpha t}\cos\omega t + Be^{\alpha t}\sin\omega t$
$a_0 + a_1 t + \cdots + a_n t^n$	$b_0 + b_1 t + \cdots + b_n t^n$

Guessing the form of a particular solution

Task: to find a *particular* solution $y_p(t)$ to:

$$y''(t) + a\,y'(t) + b\,y(t) = f(t)$$

[Table 1.2] : *Undetermined coefficients method*

Form of $f(t)$	Guess for $y_p(t)$
$ce^{\alpha t}$	$Ae^{\alpha t}$
$c\cos\omega t$ or $c\sin\omega t$	$A\cos\omega t + B\sin\omega t$
$ce^{\alpha t}\cos\omega t$ or $ce^{\alpha t}\sin\omega t$	$Ae^{\alpha t}\cos\omega t + Be^{\alpha t}\sin\omega t$
$a_0 + a_1 t + \cdots + a_n t^n$	$b_0 + b_1 t + \cdots + b_n t^n$

Substitute, simplify, and match coefficients!

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 3t$

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 3t$

- Guess $y_p(t) = A\cos 3t + B\sin 3t$
  (A and B are the 'undetermined coefficients')

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 3t$

- Guess $y_p(t) = A\cos 3t + B\sin 3t$
  (A and B are the 'undetermined coefficients')
- $y_p' = -3A\sin 3t + 3B\cos 3t$

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 3t$

- Guess $y_p(t) = A\cos 3t + B\sin 3t$
  (A and B are the 'undetermined coefficients')
- $y_p' = -3A\sin 3t + 3B\cos 3t$
- $y_p'' = -9A\cos 3t - 9B\sin 3t$

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 3t$

- Guess $y_p(t) = A\cos 3t + B\sin 3t$
  (A and B are the 'undetermined coefficients')
- $y_p' = -3A\sin 3t + 3B\cos 3t$
- $y_p'' = -9A\cos 3t - 9B\sin 3t$
- Substituting,

$$-9A\cos 3t - 9B\sin 3t + 4\left(A\cos 3t + B\sin 3t\right) = 5\cos 3t$$

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 3t$

- Guess $y_p(t) = A\cos 3t + B\sin 3t$
  (A and B are the 'undetermined coefficients')
- $y_p' = -3A\sin 3t + 3B\cos 3t$
- $y_p'' = -9A\cos 3t - 9B\sin 3t$
- Substituting,
  $$-9A\cos 3t - 9B\sin 3t + 4\left(A\cos 3t + B\sin 3t\right) = 5\cos 3t$$
- $-5A\cos 3t - 5B\sin 3t = 5\cos 3t$

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 3t$

- Guess $y_p(t) = A\cos 3t + B\sin 3t$
  (A and B are the 'undetermined coefficients')
- $y_p' = -3A\sin 3t + 3B\cos 3t$
- $y_p'' = -9A\cos 3t - 9B\sin 3t$
- Substituting,
  $$-9A\cos 3t - 9B\sin 3t + 4\left(A\cos 3t + B\sin 3t\right) = 5\cos 3t$$
- $-5A\cos 3t - 5B\sin 3t = 5\cos 3t + 0\sin 3t$
- Equate coefficients: $-5A = 5$ and $-5B = 0$

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 3t$

- Guess $y_p(t) = A\cos 3t + B\sin 3t$
  (A and B are the 'undetermined coefficients')
- $y_p' = -3A\sin 3t + 3B\cos 3t$
- $y_p'' = -9A\cos 3t - 9B\sin 3t$
- Substituting,
  $$-9A\cos 3t - 9B\sin 3t + 4\left(A\cos 3t + B\sin 3t\right) = 5\cos 3t$$
- $-5A\cos 3t - 5B\sin 3t = 5\cos 3t + 0\sin 3t$
- Equate coefficients: $-5A = 5$ and $-5B = 0$
  $\Rightarrow \quad A = -1$ and $B = 0$

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 3t$

- Guess $y_p(t) = A\cos 3t + B\sin 3t$
  (A and B are the 'undetermined coefficients')
- $y_p' = -3A\sin 3t + 3B\cos 3t$
- $y_p'' = -9A\cos 3t - 9B\sin 3t$
- Substituting,
  $$-9A\cos 3t - 9B\sin 3t + 4\left(A\cos 3t + B\sin 3t\right) = 5\cos 3t$$
- $-5A\cos 3t - 5B\sin 3t = 5\cos 3t + 0\sin 3t$
- Equate coefficients: $-5A = 5$ and $-5B = 0$
  $\Rightarrow \quad A = -1$ and $B = 0$
- In general, will have to solve simultaneous equations

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 3t$

- Guess $y_p(t) = A\cos 3t + B\sin 3t$
  (A and B are the 'undetermined coefficients')
- $y_p' = -3A\sin 3t + 3B\cos 3t$
- $y_p'' = -9A\cos 3t - 9B\sin 3t$
- Substituting,
  $$-9A\cos 3t - 9B\sin 3t + 4\left(A\cos 3t + B\sin 3t\right) = 5\cos 3t$$
- $-5A\cos 3t - 5B\sin 3t = 5\cos 3t + 0\sin 3t$
- Equate coefficients: $-5A = 5$ and $-5B = 0$
  $\Rightarrow \quad A = -1$ and $B = 0$
- In general, will have to solve simultaneous equations
- $y_p(t) = -\cos 3t$

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 3t$

- Guess $y_p(t) = A\cos 3t + B\sin 3t$
  (A and B are the 'undetermined coefficients')
- $y_p' = -3A\sin 3t + 3B\cos 3t$
- $y_p'' = -9A\cos 3t - 9B\sin 3t$
- Substituting,
  $$-9A\cos 3t - 9B\sin 3t + 4\left(A\cos 3t + B\sin 3t\right) = 5\cos 3t$$
- $-5A\cos 3t - 5B\sin 3t = 5\cos 3t + 0\sin 3t$
- Equate coefficients: $-5A = 5$ and $-5B = 0$
  $\Rightarrow \quad A = -1$ and $B = 0$
- In general, will have to solve simultaneous equations
- $y_p(t) = -\cos 3t$
- Guessing *both* sine/cosine terms is usually necessary, e.g.,
  Homework: Find y_p for $y'' + y' + 4y = 5\cos 3t$.

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 2t$

- Only change from previous example: 2 rather than 3 2min

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 2t$

- Only change from previous example: 2 rather than 3
- $y_p = A\cos 2t + B\sin 2t$
- $y_p' = -2A\sin 2t + 2B\cos 2t$ and $y_p'' = -4A\cos 2t - 4B\sin 2t$

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos\ 2t$

- Only change from previous example: 2 rather than 3
- $y_p = A\cos 2t + B\sin 2t$
- $y_p' = -2A\sin 2t + 2B\cos 2t$ and $y_p'' = -4A\cos 2t - 4B\sin 2t$
- $(-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 5\cos 2t$

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 2t$

- Only change from previous example: 2 rather than 3
- $y_p = A\cos 2t + B\sin 2t$
- $y_p' = -2A\sin 2t + 2B\cos 2t$ and $y_p'' = -4A\cos 2t - 4B\sin 2t$
- $(-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 5\cos 2t$
- $0 = 5\cos 3t$ *??????*

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos\ 2t$

- Only change from previous example: 2 rather than 3
- $y_p = A\cos 2t + B\sin 2t$
- $y_p' = -2A\sin 2t + 2B\cos 2t$ and $y_p'' = -4A\cos 2t - 4B\sin 2t$
- $(-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 5\cos 2t$
- $0 = 5\cos 3t$ *??????*
- *Problem:* guessed solution is part of the *homogeneous* solution, so when substituted into the left side it becomes zero!

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos\ 2t$

- Only change from previous example: 2 rather than 3
- $y_p = A\cos 2t + B\sin 2t$
- $y_p' = -2A\sin 2t + 2B\cos 2t$ and $y_p'' = -4A\cos 2t - 4B\sin 2t$
- $(-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 5\cos 2t$
- $0 = 5\cos 3t$ *??????*
- *Problem:* guessed solution is part of the *homogeneous* solution, so when substituted into the left side it becomes zero!
- Therefore, each such coefficient goes away $\Rightarrow$ no information!

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 2t$

- Only change from previous example: 2 rather than 3
- $y_p = A\cos 2t + B\sin 2t$
- $y_p' = -2A\sin 2t + 2B\cos 2t$ and $y_p'' = -4A\cos 2t - 4B\sin 2t$
- $(-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 5\cos 2t$
- $0 = 5\cos 3t$ *??????*
- *Problem:* guessed solution is part of the *homogeneous* solution, so when substituted into the left side it becomes zero!
- Therefore, each such coefficient goes away ⇒ no information!
- Recall multiplication by t in homogeneous case when we had difficulties in finding solution. Correct guess:

$$y_p(t) = t\,(A\cos 2t + B\sin 2t)$$

Undetermined coefficients examples

Example: Find a particular solution to $y''(t) + 4y(t) = 5\cos 2t$

- Only change from previous example: 2 rather than 3
- $y_p = A\cos 2t + B\sin 2t$
- $y_p' = -2A\sin 2t + 2B\cos 2t$ and $y_p'' = -4A\cos 2t - 4B\sin 2t$
- $(-4A\cos 2t - 4B\sin 2t) + 4(A\cos 2t + B\sin 2t) = 5\cos 2t$
- $0 = 5\cos 3t$ *??????*
- *Problem:* guessed solution is part of the *homogeneous* solution, so when substituted into the left side it becomes zero!
- Therefore, each such coefficient goes away $\Rightarrow$ no information!
- Recall multiplication by t in homogeneous case when we had difficulties in finding solution. Correct guess:

$$y_p(t) = t\,(A\cos 2t + B\sin 2t)$$

- **Homework**: complete the solution of this problem

Finding particular solution (updated)

[Table 1.2] : *Undetermined coefficients method*

Form of $f(t)$	Guess for $y_p(t)$
$ce^{\alpha t}$	$Ae^{\alpha t}$
$c\cos\omega t$ or $c\sin\omega t$	$A\cos\omega t + B\sin\omega t$
$ce^{\alpha t}\cos\omega t$ or $ce^{\alpha t}\sin\omega t$	$Ae^{\alpha t}\cos\omega t + Be^{\alpha t}\sin\omega t$
$a_0 + a_1 t + \cdots + a_n t^n$	$b_0 + b_1 t + \cdots + b_n t^n$

- If *any* part of the guessed solution is part of the homogeneous solution, multiply *everything* by the independent variable t

Finding particular solution (updated)

[Table 1.2] : *Undetermined coefficients method*

Form of $f(t)$	Guess for $y_p(t)$
$ce^{\alpha t}$	$Ae^{\alpha t}$
$c\cos\omega t$ or $c\sin\omega t$	$A\cos\omega t + B\sin\omega t$
$ce^{\alpha t}\cos\omega t$ or $ce^{\alpha t}\sin\omega t$	$Ae^{\alpha t}\cos\omega t + Be^{\alpha t}\sin\omega t$
$a_0 + a_1 t + \cdots + a_n t^n$	$b_0 + b_1 t + \cdots + b_n t^n$

- If *any* part of the guessed solution is part of the homogeneous solution, multiply *everything* by the independent variable t
- If there are *still* parts of the homogeneous solution, multiply everything *again* by t

A question

Breakout: What is the correct form for the particular solution guess for the DE

$$\frac{d^2\phi}{dx^2} - 2\frac{d\phi}{dx} + \phi = 3e^x \quad ?$$

(a) $\phi_p(x) = Ae^x$
(b) $\phi_p(x) = Axe^x$
(c) $\phi_p(x) = Ax^2e^x$
(d) $\phi_p(x) = (Ax + B)e^x$

A question

Breakout: What is the correct form for the particular solution guess for the DE

$$\frac{d^2\phi}{dx^2} - 2\frac{d\phi}{dx} + \phi = 3e^x \quad ?$$

(a) $\phi_p(x) = Ae^x$
(b) $\phi_p(x) = Axe^x$
(c) $\phi_p(x) = Ax^2e^x$
(d) $\phi_p(x) = (Ax + B)e^x$

Need to know homogeneous solution!

$$\lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad (\lambda - 1)^2 = 0$$

Repeated root of $\lambda = 1 \quad \Rightarrow e^x$

Other solution is therefore $xe^x \quad \Rightarrow \phi_h(x) = C_1e^x + C_2xe^x$

General solution to nonhomogeneous DE

- For some forms of $f(t)$, we now know how to find a *particular solution* $y_p(t)$ to

$$y''(t) + a\,y'(t) + b\,y(t) = f(t) \qquad (\dagger)$$

- What about the *general solution*?

General solution to nonhomogeneous DE

- For some forms of $f(t)$, we now know how to find a *particular solution* $y_p(t)$ to

$$y''(t) + a\,y'(t) + b\,y(t) = f(t) \qquad (\dagger)$$

- What about the *general solution*?
- Suppose you can find the full *homogeneous solution* $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$ to

$$y''(t) + a\,y'(t) + b\,y(t) = 0$$

General solution to nonhomogeneous DE

- For some forms of $f(t)$, we now know how to find a *particular solution* $y_p(t)$ to

$$y''(t) + a\,y'(t) + b\,y(t) = f(t) \qquad (\dagger)$$

- What about the *general solution*?
- Suppose you can find the full *homogeneous solution* $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$ to

$$y''(t) + a\,y'(t) + b\,y(t) = 0$$

- [Property 1.8] : $(\dagger)$ has the *general solution*

$$y(t) = y_h(t) + y_p(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$$

General solution to nonhomogeneous DE

- For some forms of $f(t)$, we now know how to find a *particular solution* $y_p(t)$ to

$$y''(t) + a\,y'(t) + b\,y(t) = f(t) \qquad (\dagger)$$

- What about the *general solution*?
- Suppose you can find the full *homogeneous solution* $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$ to

$$y''(t) + a\,y'(t) + b\,y(t) = 0$$

- [Property 1.8] : $(\dagger)$ has the *general solution*

$$y(t) = y_h(t) + y_p(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$$

- For details, see [Property 1.8] in course notes. For a brief idea of why – what happens when we substitute this into the left side of $(\dagger)$?

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

[Property 1.9] : Solving an initial value nonhomogeneous DE

(i) Find $y_h(t)$ (contains two arbitrary constants)

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

[Property 1.9] : Solving an initial value nonhomogeneous DE

(i) Find $y_h(t)$ (contains two arbitrary constants)

(ii) Find $y_p(t)$ using undetermined coefficients

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

[Property 1.9] : Solving an initial value nonhomogeneous DE

(i) Find $y_h(t)$ (contains two arbitrary constants)

(ii) Find $y_p(t)$ using undetermined coefficients

(iii) General solution $y(t) = y_h(t) + y_p(t)$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

[Property 1.9] : Solving an initial value nonhomogeneous DE

(i) Find $y_h(t)$ (contains two arbitrary constants)

(ii) Find $y_p(t)$ using undetermined coefficients

(iii) General solution $y(t) = y_h(t) + y_p(t)$

(iv) Then, *and only then*, use initial conditions to find solution to initial value problem

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\ y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t$

Form of $f(t)$	Guess for $y_p(t)$
$ce^{\alpha t}$	$Ae^{\alpha t}$
$c\cos\omega t$ or $c\sin\omega t$	$A\cos\omega t + B\sin\omega t$
$ce^{\alpha t}\cos\omega t$ or $ce^{\alpha t}\sin\omega t$	$Ae^{\alpha t}\cos\omega t + Be^{\alpha t}\sin\omega t$
$a_0 + a_1 t + \cdots + a_n t^n$	$b_0 + b_1 t + \cdots + b_n t^n$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t$
- Guess $y_p(t) = At^2 + Bt + C \quad \Rightarrow \quad y_p' = 2At + B\,,\; y_p'' = 2A$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t$
- Guess $y_p(t) = At^2 + Bt + C \quad \Rightarrow \quad y_p' = 2At + B\,,\; y_p'' = 2A$
- $2A - 4\,(2At + B) + 5\left(At^2 + Bt + C\right) = 25t^2$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\ y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t$
- Guess $y_p(t) = At^2 + Bt + C \quad \Rightarrow \quad y_p' = 2At + B\,,\ y_p'' = 2A$
- $2A - 4\,(2At + B) + 5\left(At^2 + Bt + C\right) = 25t^2$
- $(5A)\,t^2 + (-8A + 5B)\,t + (2A - 4B + 5C) = 25t^2 + 0t + 0$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t$
- Guess $y_p(t) = At^2 + Bt + C \quad \Rightarrow \quad y_p' = 2At + B\,,\; y_p'' = 2A$
- $2A - 4\,(2At + B) + 5\,\left(At^2 + Bt + C\right) = 25t^2$
- $(5A)\,t^2 + (-8A + 5B)\,t + (2A - 4B + 5C) = 25t^2 + 0t + 0$
- $A = 5,$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t$
- Guess $y_p(t) = At^2 + Bt + C \quad \Rightarrow \quad y_p' = 2At + B\,,\; y_p'' = 2A$
- $2A - 4\,(2At + B) + 5\left(At^2 + Bt + C\right) = 25t^2$
- $(5A)\,t^2 + (-8A + 5B)\,t + (2A - 4B + 5C) = 25t^2 + 0t + 0$
- $A = 5,\; B = 8,$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t$
- Guess $y_p(t) = At^2 + Bt + C \quad \Rightarrow \quad y_p' = 2At + B\,,\; y_p'' = 2A$
- $2A - 4\,(2At + B) + 5\,\left(At^2 + Bt + C\right) = 25t^2$
- $(5A)\,t^2 + (-8A + 5B)\,t + (2A - 4B + 5C) = 25t^2 + 0t + 0$
- $A = 5,\; B = 8,\; C = 22/5$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t}\cos t + C_2 e^{2t}\sin t$
- Guess $y_p(t) = At^2 + Bt + C \quad \Rightarrow \quad y_p' = 2At + B\,,\; y_p'' = 2A$
- $2A - 4\,(2At + B) + 5\,\left(At^2 + Bt + C\right) = 25t^2$
- $(5A)\,t^2 + (-8A + 5B)\,t + (2A - 4B + 5C) = 25t^2 + 0t + 0$
- $A = 5,\; B = 8,\; C = 22/5 \quad \Rightarrow \quad y_p(t) = 5t^2 + 8t + 22/5$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t}\cos t + C_2 e^{2t}\sin t$
- Guess $y_p(t) = At^2 + Bt + C \quad \Rightarrow \quad y_p' = 2At + B\,,\; y_p'' = 2A$
- $2A - 4\,(2At + B) + 5\left(At^2 + Bt + C\right) = 25t^2$
- $(5A)\,t^2 + (-8A + 5B)\,t + (2A - 4B + 5C) = 25t^2 + 0t + 0$
- $A = 5,\; B = 8,\; C = 22/5 \quad \Rightarrow \quad y_p(t) = 5t^2 + 8t + 22/5$
- $y(t) = C_1 e^{2t}\cos t + C_2 e^{2t}\sin t + 5t^2 + 8t + 22/5$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\ y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t}\cos t + C_2 e^{2t}\sin t$
- Guess $y_p(t) = At^2 + Bt + C \quad \Rightarrow \quad y_p' = 2At + B\,,\ y_p'' = 2A$
- $2A - 4\,(2At + B) + 5\left(At^2 + Bt + C\right) = 25t^2$
- $(5A)\,t^2 + (-8A + 5B)\,t + (2A - 4B + 5C) = 25t^2 + 0t + 0$
- $A = 5,\ B = 8,\ C = 22/5 \quad \Rightarrow \quad y_p(t) = 5t^2 + 8t + 22/5$
- $y(t) = C_1 e^{2t}\cos t + C_2 e^{2t}\sin t + 5t^2 + 8t + 22/5$
- $y(0) = 0 \quad \Rightarrow \quad C_1 + 22/5 = 0 \Rightarrow C_1 = -22/5$

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t$
- Guess $y_p(t) = At^2 + Bt + C \quad \Rightarrow \quad y_p' = 2At + B\,,\; y_p'' = 2A$
- $2A - 4\,(2At + B) + 5\left(At^2 + Bt + C\right) = 25t^2$
- $(5A)\,t^2 + (-8A + 5B)\,t + (2A - 4B + 5C) = 25t^2 + 0t + 0$
- $A = 5,\; B = 8,\; C = 22/5 \quad \Rightarrow \quad y_p(t) = 5t^2 + 8t + 22/5$
- $y(t) = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t + 5t^2 + 8t + 22/5$
- $y(0) = 0 \quad \Rightarrow \quad C_1 + 22/5 = 0 \Rightarrow C_1 = -22/5$
- Complete as homework (should get $C_2 = 9/5$)

An example

Example: Solve the initial value problem

$$y''(t) - 4\,y'(t) + 5\,y(t) = 25t^2 \quad ; \quad y(0) = 0\,,\; y'(0) = 1$$

- $y'' - 4y' + 5 = 0 \quad \Rightarrow \quad \lambda^2 - 4\lambda + 5 = 0$
- $\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i \quad \Rightarrow \quad y_h(t) = C_1 e^{2t}\cos t + C_2 e^{2t}\sin t$
- Guess $y_p(t) = At^2 + Bt + C \quad \Rightarrow \quad y_p' = 2At + B\,,\; y_p'' = 2A$
- $2A - 4\,(2At + B) + 5\left(At^2 + Bt + C\right) = 25t^2$
- $(5A)\,t^2 + (-8A + 5B)\,t + (2A - 4B + 5C) = 25t^2 + 0t + 0$
- $A = 5,\; B = 8,\; C = 22/5 \quad \Rightarrow \quad y_p(t) = 5t^2 + 8t + 22/5$
- $y(t) = C_1 e^{2t}\cos t + C_2 e^{2t}\sin t + 5t^2 + 8t + 22/5$
- $y(0) = 0 \quad \Rightarrow \quad C_1 + 22/5 = 0 \Rightarrow C_1 = -22/5$
- Complete as homework (should get $C_2 = 9/5$)
- $y(t) = -\frac{22}{5} e^{2t}\cos t + \frac{9}{5} e^{2t}\sin t + 5t^2 + 8t + \frac{22}{5}$

Undamped, forced, mechanical vibrations

[Example 1.32] What is the long-term behaviour of

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\sin\omega t \qquad ?$$

Undamped, forced, mechanical vibrations

[Example 1.32] What is the long-term behaviour of

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\sin\omega t \qquad ?$$

- Homogeneous: $x'' + \frac{k}{m}x = 0 \quad \Rightarrow \quad \lambda^2 + \frac{k}{m} = 0$

Undamped, forced, mechanical vibrations

[Example 1.32] What is the long-term behaviour of

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\sin\omega t \qquad ?$$

- Homogeneous: $x'' + \frac{k}{m}x = 0 \quad \Rightarrow \quad \lambda^2 + \frac{k}{m} = 0$
- If $\omega_0 := \sqrt{\frac{k}{m}}$, $\lambda = \pm i\,\omega_0 \Rightarrow x_h(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t$

Undamped, forced, mechanical vibrations

[Example 1.32] What is the long-term behaviour of

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\sin\omega t \qquad ?$$

- Homogeneous: $x'' + \frac{k}{m}x = 0 \quad \Rightarrow \quad \lambda^2 + \frac{k}{m} = 0$
- If $\omega_0 := \sqrt{\frac{k}{m}}$, $\lambda = \pm i\,\omega_0 \Rightarrow x_h(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$
- $x_p(t) = A\cos\omega t + B\sin\omega t$

Undamped, forced, mechanical vibrations

[Example 1.32] What is the long-term behaviour of

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\sin \omega t \qquad ?$$

- Homogeneous: $x'' + \frac{k}{m}x = 0 \quad \Rightarrow \quad \lambda^2 + \frac{k}{m} = 0$
- If $\omega_0 := \sqrt{\frac{k}{m}}$, $\lambda = \pm i\,\omega_0 \Rightarrow x_h(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$
- $x_p(t) = A \cos \omega t + B \sin \omega t$
- $x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + A \cos \omega t + B \sin \omega t$

Undamped, forced, mechanical vibrations

[Example 1.32] What is the long-term behaviour of

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\sin\omega t \qquad ?$$

- Homogeneous: $x'' + \frac{k}{m}x = 0 \quad \Rightarrow \quad \lambda^2 + \frac{k}{m} = 0$
- If $\omega_0 := \sqrt{\frac{k}{m}}$, $\lambda = \pm i\,\omega_0 \Rightarrow x_h(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t$
- $x_p(t) = A\cos\omega t + B\sin\omega t$ *(if $\omega \neq \omega_0$)*
- $x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + A\cos\omega t + B\sin\omega t$

Undamped, forced, mechanical vibrations

[Example 1.32] What is the long-term behaviour of

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\sin\omega t \qquad ?$$

- Homogeneous: $x'' + \frac{k}{m}x = 0 \quad \Rightarrow \quad \lambda^2 + \frac{k}{m} = 0$
- If $\omega_0 := \sqrt{\frac{k}{m}}$, $\lambda = \pm i\,\omega_0 \Rightarrow x_h(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t$
- $x_p(t) = A\cos\omega t + B\sin\omega t$ *(if $\omega \neq \omega_0$)*
- $x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + A\cos\omega t + B\sin\omega t$
- *If $\omega = \omega_0$*, then $x_p(t) = t\,(A\cos\omega_0 t + B\sin\omega_0 t)$

Undamped, forced, mechanical vibrations

[Example 1.32] What is the long-term behaviour of

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\sin\omega t \qquad ?$$

- Homogeneous: $x'' + \frac{k}{m}x = 0 \quad \Rightarrow \quad \lambda^2 + \frac{k}{m} = 0$
- If $\omega_0 := \sqrt{\frac{k}{m}}$, $\lambda = \pm i\,\omega_0 \Rightarrow x_h(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t$
- $x_p(t) = A\cos\omega t + B\sin\omega t$ *(if $\omega \neq \omega_0$)*
- $x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + A\cos\omega t + B\sin\omega t$
- *If* $\omega = \omega_0$, then $x_p(t) = t\,(A\cos\omega_0 t + B\sin\omega_0 t)$
- $x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + t\,(A\cos\omega_0 t + B\sin\omega_0 t)$

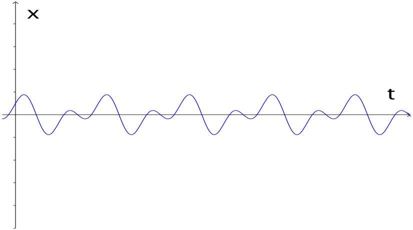
Undamped, forced, mechanical vibrations

[Example 1.32] What is the long-term behaviour of

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\sin\omega t \qquad ?$$

$\omega \neq \omega_0$

$$\begin{aligned} x(t) &= C_1\cos\omega_0 t + C_2\sin\omega_0 t \\ &\quad + (A\cos\omega t + B\sin\omega t) \end{aligned}$$



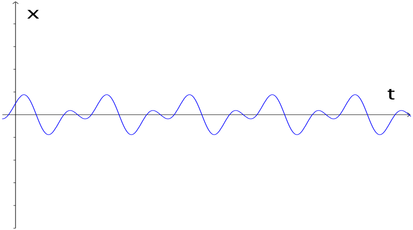
Undamped, forced, mechanical vibrations

[Example 1.32] What is the long-term behaviour of

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\sin\omega t \qquad ?$$

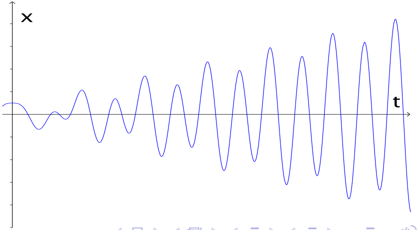
$\omega \neq \omega_0$

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + (A\cos\omega t + B\sin\omega t)$$



$\omega = \omega_0$

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + t\,(A\cos\omega_0 t + B\sin\omega_0 t)$$



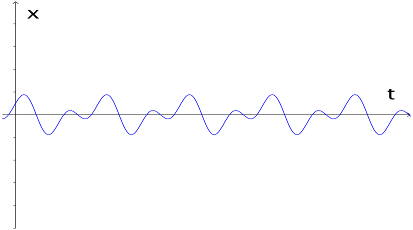
Undamped, forced, mechanical vibrations

[Example 1.32] What is the long-term behaviour of

$$x''(t) + \frac{k}{m}x(t) = \frac{F_0}{m}\sin \omega t \qquad ?$$

$\omega \neq \omega_0$

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + (A \cos \omega t + B \sin \omega t)$$



$\omega = \omega_0$ RESONANCE

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + t\,(A \cos \omega_0 t + B \sin \omega_0 t)$$